Computing time-consistent Markov policies for quasi-hyperbolic consumers under uncertainty*

Łukasz Balbus† Kevin Reffett‡ Łukasz Woźny§

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Abstract

We study the question of existence and computation of time-consistent Markov policies of quasi-hyperbolic consumers under a stochastic transition technology and borrowing constraints. Under standard assumptions on preferences, as well as a mild geometric condition on a transition probabilities, we prove existence of the greatest and the least time-consistent policies, and provide conditions for (Lipschitz) continuous and monotone equilibria. We show how our methods extend the results in Harris and Laibson (2001). We present a simple approximation scheme for extremal equilibrium, and provide some monotone equilibrium comparative statics results in the model's deep parameters.

1 Introduction

The problem of dynamic inconsistency first introduced in Strotz (1956), and further developed in Phelps and Pollak (1968) or Peleg and Yaari (1973), has played an increasingly important role in many fields in economics. Recent work where this problem appears includes papers on such diverse topics as: optimal consumption/savings decisions, role of liquidity constraints in dynamic asset markets, the behavioral foundations of economic choice, the role of commitment devices in dynamic models of self-control, the design of dynamic time-consistent environmental policies, and various papers studying the welfare implications of public policy in dynamic models.[1] The classical

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†Institute of Mathematics, Wrocław University of Technology, 50-370 Wrocław, Poland.

‡Department of Economics, Arizona State University, USA.

§Department of Theoretical and Applied Economics, Warsaw School of Economics, Warsaw, Poland. Address: al. Niepodległości 162, 02-554 Warszawa, Poland. E-mail: lukasz.wozny@sgh.waw.pl.

[1] For a small sampling of this work, see the papers of O'Donoghue and Rabin (1999a,b), Laibson (1997) and Angeletos, Laibson, Repetto, Tobacman, and Weinberg (2001), or

toolkit for analyzing such ”time” consistency problems was first proposed by Strotz (1956), and emphasized the language of recursive decision theory. However, as observed by many researchers in subsequent discussions (e.g., Peleg and Yaari (1973) and Bernheim and Ray (1986)), such optimal dynamically (or time) consistent plans need not exist, let alone be simple to characterize and/or compute.

One key reason for this fact lies in the almost inherit presence of discontinuities in intertemporal preferences (or dynamic equilibrium) that arises naturally in such problems in recursive decision theory. The source of the lack of continuity is generated by the lack of commitment between the current ”versions” of the decisionmaker, and all her continuation ”selves”. For example, from a decision theoretic perspective, when a ”current” decision maker is indifferent between some alternatives in the future, that same decision maker can still strictly prefer such an alternative in advance and be willing to commit, yets lacks access to a reasonable ”commitment device” that would impose discipline on her future ”selves” when tomorrow arrives. Due to this discontinuity, the optimal level of ”commitment” may be nonexistent (see, for example, Caplin and Leahy (2006)), and the dynamic maximization problem can be poorly defined.

As a way of circumventing these problems, Peleg and Yaari (1973) proposed a dynamic game interpretation of the time-consistency problem. More specifically, in this view of the problem, one envisions the decisionmaker playing a dynamic game between one’s current self, and each of her future ”selves”, with the solution concept in the game being a subgame-perfect Nash equilibrium (SPNE, henceforth). From a decision theoretic viewpoint, although optimal time-consistent policies and SPNE are linked, i.e. every optimal time-consistent plan is a SPNE of the appropriate game, the converse is not in general true. This latter fact is due, in part, to the dynamic decision theoretic approach being proposed itself, where future ties are broken in favor of a current self, and that observation is not necessarily true for a SPNE of a dynamic game. Additionally, the set of SPNE may be large, and most importantly, not necessarily closed; hence, an optimal SPNE (corresponding to an optimal time-consistent policy) may simply not exist. This latter issue has become the central concern of a now broad literature that has emerged since the pioneering work of Peleg and Yaari (1973). Moreover, even if the question of existence of SPNE is resolved, the equilibrium existence in the particular class of functions, namely Stationary Markov Nash Equilibria (henceforth, SMNE) is still not guaranteed (see Bernheim and Ray (1986) and Leininger (1986)). The work of Maskin and Tirole (2001) provides an extensive set of motivations for why one might be interested in concentrating on SMNE, as opposed to SPNE. Finally, the recent em-

Eisenhauer and Ventura (2006) for an empirical evidence supporting quasi-hyperbolic preferences hypothesis, among others.

phasis on numerical simulations in applied work studying models with time consistency issues also provides additional reasons for being interested in SMNE.

In this paper, we develop a classical Strotz (1956) recursive approach to studying equilibrium in the Phelps and Pollak (1968) game theoretic representation of the problem, with an emphasis on developing constructive methods for characterizing SMNE (as well as computing them). That is, we seek conditions under which simple stable iterative numerical algorithms can be developed that both (i) characterize the existence of SMNE from a theoretical perspective, as well (ii) provide explicit and accurate algorithms for computing them. From the perspective of the existence question, our paper is very closely related to the important papers of Bernheim and Ray (1986) or Harris and Laibson (2001), where the authors add noise of invariant support in an effort to develop conditions that guarantee the existence of an time-consistent policy of locally bounded variation and/or Lipschitz for sufficiently small amount of hyperbolic discount factor. But what is critical in understanding the difference between the approaches in this literature, and those in the present paper, is that the methods we propose do not rely on so-called "generalized Euler equation" methods (as, for example, in Harris and Laibson). In the generalized Euler equation approach, the question of existence and characterization is closely related to a generalized Euler inequality method that appeals to the calculus of bounded variation for characterizing the structure of SMNE. In our paper, we propose a very different approach, one based upon a value function method (in the spirit of "promised utility methods"), but defined in spaces of *functions,* as opposed to spaces of correspondences in the promised utility literature. Our methods link the stochastic game studied in Harris and Laibson (2001), with a recursive or value function methods suggested by Strotz (1956) (and further developed by Caplin and Leahy (2006)), and therefore nicely complement all of these existing literatures.

More specifically, under standard assumptions on preferences and certain geometric condition on a transition probability, using our value iteration approach, we are able to show existence of the greatest and the least time-consistent policies, provide characterizations of their (Lipschitz) continuity and monotonicity properties, and, therefore, extend the Harris and Laibson (2001) results to *all* discount factors. Further, an important part of our results is that we can characterize the set of *all* values corresponding to time-consistent policies. In particular, we show this set is a countably chain complete poset. Finally, and equally as important, our methods allow us to obtain in a natural way the existence of both the greatest and the least value functions associated with SMNE, as well as least and greatest SMNE pure strategies that sustain each of these values (hence existence of an optimal, time-consistent decision policy is established). The fact that our methods emphasize *both* the computation of values and pure strategies is of central importance in our work.

We then turn to the question of computation of SMNE, as well as equilibrium comparative statics on the deep parameters of the game. We are able to construct an simple approximation scheme relative to the set of SMNE, as well as conduct monotone comparative statics on the extremal time-consistent SMNE policies with respect to the model parameters. The comparative statics and approximation results are important for applied research in the field. For example, in Sorger (2004), he proposes settings under which any twice continuously differentiable function can be supported as a policy of a time consistent hyperbolic consumer. This result can be subsequently linked to a Gong, Smith, and Zou (2007) text, showing that a hyperbolic discounting is not observationally equivalent to exponential discounting. That is, it is always possible to calibrate an exponential model so that it predicts the same level of consumption as a hyperbolic model. However, the two models have *radically different* comparative statics. Hence, our approach allows us to sort out the exact nature of this question, and provide theoretical monotone comparative statics results based on the equilibrium set of the stochastic game itself. Such a result can clarify empirical questions that are asked by applied researchers.

The rest of the paper is organized as follows. In section 2 we discuss in details the related literature, while in section 3 we present a general fixed point result counterparting Tarski (1955) existence and Veinott (1992) comparative statics theorems to $\sigma-$complete lattices and countably chain complete posets. In section 4 we specify model, its assumptions and state our main theorems. Finally section 5 concludes by discussing how our results can be potentially used for a study of different approaches or axiomatizations of time-consistent problems.

2 Related Literature

It is important to remember that non-existence and multiplicity problems related to the class of games we study have constituted a significant challenge for applied economists who sought to study models where such dynamic consistency failures play a key role. They have been equally as challenging for researchers that seek to identify tractable numerical approaches to computing SMNE in these (and related) dynamic games (e.g., see the discussion in Krusell and Smith (2003) or Judd (2004)). On the one hand, Krusell, Kuruscu, and Smith (2002a) propose a generalized Euler equation method for a version of a hyperbolic discounting consumer and obtain explicit solution for logarithmic utility and Cobb-Douglas production examples. But this is an example. On the other hand, in Judd (2004), he uses generalized Euler equation approach to analyze smooth time-consistent policies and proposes a perturbation method for calculating them. The problem here is providing conditions under which at any point in the state space the generalized Euler equations represent a *sufficient* first order theory for an agents value function in the equilibrium of the game.[2] Concentrating on non-smooth policies, Krusell and Smith (2003) define a step function equilibrium and show its existence and resulting indeterminacy of steady state capital levels. Further, in a deterministic setting general existence result of optimal policies under quasi-geometric discounting can be provided using techniques proposed by Goldman (1980) for finite horizon economies, by Harris (1985) for infinite horizon or by Feinberg and Shwartz (1995) in the generalized discounting setting.

Summarizing, from a technical point of view, tools used to show existence and characterize Markovian policies are wide and motivated by specific applications or problems under study. Still the general framework for studying (analytically and numerically) of (possibly nonsmooth) SMNE is missing. To circumvent some of these mentioned predicaments in a unified setup authors also added noise to the decision problems or relevant dynamic games. Specifically, in a *(recursive) decision approach*, by adding noise (making payoff discountinities negligible) Caplin and Leahy (2006) prove existence of recursively optimal plan for a finite horizon decision problem and general utility functions. Similarly Bernheim and Ray (1986) show that by adding enough noise to the dynamic game (to smooth discontinuities away) existence of SMNE is guaranteed. Such *stochastic game approach* was later developed by Harris and Laibson (2001) who characterize the set of smooth SMNE by (generalized) first order conditions.

It is worth mentioning that authors have also analyzed optimal but not necessarily time-consistent policies. For infinite horizon decision problems Kydland and Prescott (1980, henceforth KP) notice that the state space of an appropriately defined value function must incorporate some pseudo-state variables like Lagrange multipliers for the problem (of finding optimal policies) to be recursive. KP method is linked to the Abreu, Pearce, and Stacchetti (1990, henceforth APS) type arguments. Specifically by adding appropriate noise to the time-consistency game characterization of all sequential equilibria using KP/APS methods can be offered. This approach is undertaken by Bernheim, Ray, and Yeltekin (1999). They analyze our problem using KP/APS type arguments. Specifically they consider a set of (bounded) values for (sequential) subgame perfect equilibria in a Phelps and

[2] We should elaborate on this point. In a generalized Euler equation method, on an open set of any point in the state space, we can always construct an local linearization (in a space of functions) that might be valid as a linear approximation to the function that satisfies the functional equation near that point; the problem is showing the Euler equation is necessary and sufficient on that open set. For the later claim to be true, you must know the value function in the equilibrium of the game is concave. In our method, such a local expansion will be valid; but then, we do not need the generalized Euler equation to compute the models equilibrium.

Pollak (1968) self-game and analyze all subsets of such values. Later they construct a monotone (under set inclusion) operator on this set and numerically analyze its largest fixed point. Using this method they show existence of a sequential time-consistent policy and use it to analyze self-control in the context of a low asset trap.

Finally, the literature on self-control is larger than on a specific problem of time-consistency, including papers specifying preferences over menus allowing for temptation. That is, instead of taking a preference change as a primitive of the model, economist introduce preferences over menus which are time-consistent (i.e. do not change over time) but still allow for modeling of self-control (by introducing so-called set-betweenness axiom[3]).

Specifically Gul and Pesendorfer (2001) (GP, henceforth) and Dekel, Lipman, and Rustichini (2001) (DLR, henceforth) consider a general model of preferences over menus (lotteries), from which choice is made at a later date and show that preferences over menus can be used to identify an agent's subjective beliefs regarding her future tastes and behavior. They explicitly model a cost of tomorrow's temptation as a difference between tomorrow optimal decision and a current tempted decision[4]. GP introduce also an *overwhelming temptation preferences* or *Strotz representation* where the future decision are always made according to the tempted preferences, which is exactly the case in our quasi-hyperbolic discounting problem. For application of GP see Krusell, Kuruscu, and Smith (2002b) used for studying asset pricing puzzle.

Actually, there are more links between (stochastic) game methods used in this paper and the preference approach discussed above. Here we refer the reader to the paper of Benabou and Pycia (2002) who represent GP preferences by outcomes of the two-period game of control between a "planner" and a "doer". Also Fudenberg and Levine (2006) paper presents a stochastic game between planner and a sequence of myopic doers. Doers choose actions and planer their costs. They show the strategies and outcomes of their game are equivalent to solutions of a "planner" maximization problem under incentive compatible constraints. Fudenberg and Levine (2006) also discusses relation between their game and GP preference representation. Hence a natural question, on applicability of our constructive (stochastic game or stochastic decision problem) methods to the Fudenberg and Levine (2006) or Benabou and Pycia (2002) game and hence GP or DLR representation, arise. This becomes especially important in the view of Dekel and Lipman (2010) *random Strotz representation*[5], where decision from the menu is con-

[3]More recently Dekel, Lipman, and Rustichini (2009) propose alternative axiomatization of self-control preferences but still leading to possible different choice representation.

[4]See also Gul and Pesendorfer (2004) for recursive temptation driven preferences in a dynamic setting. In their section 6 they use such preferences to analyze a dynamic model of temptation driven preference in a stochastic economy.

[5]Dekel and Lipman (2010) also show that random Strotz preferences can represent GP

strained to the actions incentive compatible with (tempted) doer, but where the preferences of the doer are drawn from some probability distribution. We discuss applicability of our methods for studying such problems in section 5 of our paper.

Finally let us note that quasi-hyperbolic discounting problem is linked to a problem of altruism towards successive generations (see Saez-Marti and Weibull (2005) for formal results). This link can be also seen through a technical perspective, where the stochastic games methods (see Balbus, Reffett, and Woźny (2010)) can be applied for both quasi-hyperbolic discounting and intergenerational altruism (see Balbus, Reffett, and Woźny (2009)) models.

3 Preliminary result

We begin by discussing a new result that is essential in our subsequent arguments in this paper. The theorem is related to an important existence result concerning the fixed points of monotone transformations of complete lattices due to Tarski (1955), as well as a well-known fixed point comparative statics result due to Veinott (1992).[6] The result concerns the existence of fixed point comparative statics in parameterized monotone problems in chain complete partially ordered sets. We begin with a few important definitions.[7]

Definition 3.1 *A function $F : X \to X$ is sup-preserving (monotonically-sup-preserving), if for any sequence (monotone sequence) $\{x_n\}_{n=0}^{\infty}$ we have: $F(\bigvee x_n) = \bigvee F(x_n)$. We define inf-preserving and (monotonically inf-preserving) functions analogously. F is said to be sup-inf-preserving (monotonically-sup-inf-preserving) if and only if, it is both sup-preserving and inf-preserving (monotonically-sup and monotonically-inf-preserving).*

We should mention that a monotonically sup/inf preserving map is often referred to as *order continuous* in the literature (e.g., Dugundji and Granas (1982), p.15). The condition is closely related to the idea of a continuous function in the Scott topology. This property plays an essential role in the computation of fixed point for isotone maps in countably chain complete partially ordered sets. Additionally, the idea of an "sup/inf" preserving map is essentially a generalization of the idea of order continuity to $\sigma-$complete lattices. This concept is also directly related to the idea of an *sequential order continuous* operator developed in (Vulikh 1967, p.27).

We now state the general theorem characterizing fixed point structure of a parameterized monotone self map defined on a σ-complete lattice (or a countably chain complete partially ordered set), with an emphasis on a fixed

preferences.

[6]See also Topkis (1998), Theorem 2.5.2.

[7]Additional mathematical definitions relating to partially ordered sets and lattices are found in the appendix.

point comparative statics result concerning the monotonicity of extremal fixed point selections in its parameters. Its proof is technical and can be found in an appendix.

Theorem 3.1 *Let $F : X \times T \to X$ be an parameterized monotone increasing operator with T a poset, X a $\sigma-$complete lattice (respectively, a countably chain complete poset) with the greatest and least elements. If for any $t \in T$, the function $F(\cdot, t)$ is* sup-inf-preserving *(respectively, monotonically-sup-inf preserving), then the fixed point set of $F(\cdot, t)$, denoted by $\Phi(t)$, is a σ-complete lattice (respectively, a countably chain complete poset). Moreover, the least and greatest fixed point selections $t \to \underline{\Phi}(t) := \wedge\ \Phi(t)$ and $t \to \overline{\Phi}(t) := \vee\ \Phi(t)$ are isotone.*

Per the question of existence, the theorem is related to those found in Tarski (1955) and Markowsky (1976), as well as a theorem pertaining to the computation of fixed found in Vulikh (1967) and Dugundji and Granas (1982). For example, relative to the first two papers, in our setting, although the existence of least and greatest fixed point could be inferred from their results, Theorem 3.1 also provides a characterization of the fixed point set $\Phi(t)$ (e.g., it is either σ-complete valued or countable chain complete poset valued). This later result is new. Now, obviously, weakening of conditions in previous work does come at a cost (namely, for existence, we need an additional assumption of order continuity). Further, the converse results in Davis (1955) (e.g., theorem 2) and Markowsky (1976) (e.g., theorem 11) provide answers to why our continuity assumptions are needed for sufficiency. Relative to the latter two papers, we provide a characterization of computable fixed point comparative statics (e.g., compare our results to Vulikh (1967) (Lemma XII.2.1) and Dugundji and Granas (1982) (Theorem 4.2).

It is also worth mentioning that we generalize result of Veinott (1992) in some important directions. First, as mentioned before, $\Phi(t)$ is now only $\sigma-$complete (resp, countably chain complete) valued, but we require much less structure on the underlying domain of our operators (as the expense, though, of added continuity conditions). Second, as both the top and bottom elements of $\Phi(t)$ are increasing selections, the correspondence $\Phi(t)$ is actually directed upward and directed downward (hence, ascending in the "weak induced set order"). This is also true, for example for the complete lattice case studied in Veinott (1992).

4 Model and main results

In the environment we study, we envision an individual decisionmaker to be a sequence of "selves" indexed in discrete time $t = 0, 1, \ldots$. For a given state $x_t \in S$ (where $S = [0, \overline{S}]$ or $S = [0, \infty)$), the "self t " chooses a consumption

$c_t \in [0, x_t]$, and leaves $x_t - c_t$ as an investment for future "selves". As in effect, we rule out borrowing; also we interpret the asset as a productive one, and refer to it as capital.[8] These choices, together with current state x_t, determine a transition probability $Q(dx_{t+1}|x_t - c_t, x_t)$ of a next period state.

Self t preferences are represented by a utility function given by:

$$u(c_t) + \beta E_t \sum_{i=t+1}^{\infty} \delta^{i-t} u(c_i), \tag{1}$$

where $1 \geq \beta > 0$ and $1 > \delta \geq 0$, u is a instantaneous utility function and expectations E_t are taken with respect to a realization of a random variable x_i drawn each period from a transition distribution Q. Under some continuity assumptions on u and Q (to be specified later), we can define a Markovian equilibrium pure strategy to be an $h \in \mathcal{H}$ where $\mathcal{H} = \{h : S \to S | 0 \leq h(x) \leq x$ bounded and Borel measurable$\}$ that is time-consistent for the quasi-hyperbolic consumer. That is, if h satisfies the following functional equation:

$$h(x) \in \arg\max_{c \in [0,x]} u(c) + \beta\delta \int_S V_h(y) Q(dy|x - c, x), \tag{2}$$

where $V_h : S \to \mathbb{R}$ is a continuation value function for the household of "future" selves following a stationary policy h from tomorrow on. The value in the Markovian equilibrium for the future selves, therefore, must solve the following additional functional equation in the continuation given as follows:

$$V_h(x) = u(h(x)) + \delta \int_S V_h(y) Q(dy|x - h(x), x). \tag{3}$$

Therefore, if we define the value function for the self t to be:

$$W_h(x) := u(h(x)) + \beta\delta \int_S V_h(y) Q(dy|x - h(x), x),$$

one obtains the relation

$$V_h(x) = \frac{1}{\beta} W_h(x) - \frac{1-\beta}{\beta} u(h(x)). \tag{4}$$

Following Harris and Laibson (2001), based on equation 4, we can define an operator whose fixed points, say V^*, corresponds to values for some *time-consistent* Markov policies. To study this problem, we need to make some assumptions on the primitive data of the game to use our parameterized fixed point results in section 3 to characterize the set of Markovian equilibrium in this stochastic game. Along these lines, we make the following assumptions:

[8] We can introduce assets and borrowing into the model also, but at the expense of having to decentralize equilibrium under some meaningful price system. With stochastic production, this can be a difficult problem. We leave this question for future work.

Assumption 1 *Let us assume:*

- $u : S \to \mathbb{R}_+$ *is continuous, increasing and strictly concave with* $u(0) = 0$ *and* $u(\cdot) \leq \overline{u}$,
- *for any* $x, i \in S$ *let* $Q(\cdot|i,x) = g_0(i)\delta_0(\cdot) + \sum_{j=1}^{J} g_j(i)\lambda_j(\cdot|x)$,
- $(\forall j = 1, \dots, J)\, g_j : S \to [0,1]$ *is continuous, increasing and concave with* $g_j(0) = 0$ *and* $\sum_{j=0}^{J} g_j(i) = 1$ *for all* i,
- δ_0 *is a delta Dirac measure concentrated at point* 0*, while* $(\forall j = 1, \dots, J)\, \lambda_j(\cdot|x)$ *is a Borel transition distribution on* S *for any* $x \in S$.

First, our assumptions on preferences are completely standard. Here, we only mention the imposition of strict concavity of a utility function in these assumptions allows us to restrict attention to single valued best replies in the definition 2, and hence we can study the fixed points of a single valued operator whose fixed points generate corresponding equilibrium values and corresponding policies in the game. It bears mentioning that a careful reading of the proof of our main existence theorem below (e.g., Theorem 4.1) indicates this assumption can be weakened, as under our assumptions we can always just work with increasing selections from a best response correspondence (not necessarily unique valued best replies).

Second, our assumption on a transition probability requires a few remarks. Q is defined as a convex combination of J measures λ_j and one δ_0 Dirac. Hence, with probability $1 - \sum_{j=1}^{J} g_j(i)$, the next period wealth (or capital) is zero, and with probability $g_j(i)$ it is drawn from λ_j. Observe also that we separate decision i and state variables x in Q, i.e. λ_j are not dependent on a decision i. Our mixing condition on the stochastic transitions in the game are somewhat common in the literature, first introduced by Amir (1996), but later developed by Nowak (2003) and Balbus, Reffett, and Woźny (2009) for games with a similar structure as those studied in this paper, and for a very general class of stochastic supermodular games in Balbus, Reffett, and Woźny (2010). Even this assumption can be weakened per questions of existence (e.g., per the application of the APS procedure developed in Balbus, Reffett, and Woźny (2010)), but at the cost of computing *both* equilibrium values and pure strategy Markovian equilibrium.

Finally, our motivation and approach is closely linked to that of Harris and Laibson (2001), so it is useful to compare our assumptions, and theirs. First, we do not impose twice continuous differentiability nor strict monotonicity of a utility function as Harris and Laibson do. Second, as opposed to Harris and Laibson, although we bound our utility between 0 and $\overline{u}$, we do not bound its risk aversion. This latter assumption is critical for any generalized implicit function approach (or calculus of bounded variation). Third, as far as transition probability is concerned, our model generates more sources of noise than Harris and Laibson (2001), as in our case, not only is labor income random, but wealth (or capital) is also draw from Q. Finally, we do not require that Q has a density, let alone impose conditions on its degree of smoothness.

4.1 Existence

Let $\mathcal{V}$ be a space of bounded (by 0 and $\frac{\overline{u}}{1-\delta}$), Borel measurable, real valued functions on S, with $V(0) = 0$ equipped with a pointwise partial order. For a given value $V \in \mathcal{V}$ construct an operator T by:

$$TV(x) = \frac{1}{\beta} AV(x) - \frac{1-\beta}{\beta} u(BV(x)), \tag{5}$$

where the pair of operators A and B defined on space $\mathcal{V}$ are given by:

$$AV(x) = \max_{c\in[0,x]} \left\{ u(c) + \beta\delta \int_S V(y) Q(dy|x-c, x) \right\}, \tag{6}$$

$$BV(x) = \arg\max_{c\in[0,x]} \left\{ u(c) + \beta\delta \int_S V(y) Q(dy|x-c, x) \right\}. \tag{7}$$

Notice, in the above, we have defined the operator B to map between candidates for equilibrium values $\mathcal{V}$ to spaces of pure strategy best replies $\mathcal{H}$. So in effect, we have a pair of operator equation we need to solve to construct equilibrium values $V^* \in \mathcal{V}$.

Surely T maps $\mathcal{V}$ into itself. Further, for any fixed point V^* of an operator T, this value function corresponds to a stationary, time-consistent Markov policy $h^* = BV^* \in \mathcal{H}$. To see this latter claim, we first prove an important lemma. Equip the space of pure strategies $\mathcal{H}$ with a pointwise partial order. In this case, we obtain our first lemma.

Lemma 4.1 *Let assumption 1 hold then $A : \mathcal{V} \to \mathcal{V}$ is increasing and $B : \mathcal{V} \to \mathcal{H}$ is decreasing. Moreover, T is increasing and monotonically-sup-inf-preserving.*

Proof: A is increasing by definition. To see monotonicity of B, consider a function

$$G(c, x, V) = u(c) + \beta\delta \sum_{j=1}^{J} g_j(x-c) \int_S V(y) \lambda_j(dy|x).$$

Then for any $V \in \mathcal{V}$ and $x \in S$, the function $G(\cdot, x, V)$ is supermodular.

Moreover, $(c, V) \to g_j(x-c) \int_S V(y)\lambda_j(dy|x)$ has decreasing differences. To see this fact, observe we have the following inequalities:

$$\begin{aligned} & [g_j(x-c_2) - g_j(x-c_1)] \int_S V_2(y)\lambda_j(dy|x) \\ \leq \; & [g_j(x-c_2) - g_j(x-c_1)] \int_S V_1(y)\lambda_j(dy|x) \end{aligned}$$

where $V_2 \geq V_1$ and $c_2 \geq c_1$. Therefore, for any $x \in S$, the function $(c, V) \to G(c, x, V)$ has decreasing differences on $[0, x] \times \mathcal{V}$. Since $[0, x]$ is a lattice and $\mathcal{V}$ is poset, we obtain by Topkis (1978) theorem that the (unique) best reply $BR(V)(x) = \arg\max_{c \in [0,x]} G(c, x, V)$ is decreasing on $\mathcal{V}$. Since A is increasing and B decreasing, by definition of T, we conclude that T is increasing.

We now show that T is monotonically-sup-inf preserving. Let $\{V_n\}_{n=1}^{\infty} \subset \mathcal{V}$ be increasing sequence in the natural product order. Let $V_n \to V$ pointwise. Clearly, $V(s) = \sup_{n \in \mathbf{N}} V_n(s)$. We need to show $\lim_{n \to \infty} T(V_n) = T(V)$. By Lebesgue Dominating Theorem, we immediately obtain:

$$\int_S V_n(y) \lambda_j(dy|x) \to \int_S V(y) \lambda_j(dy|x) \quad \text{as } n \to \infty,$$

for all j and x. For fixed x, let $c_n := B(V_n)(x)$. Since c_n belongs to compact set $[0, x]$, without loss of generality, let us assume $c_n \to c_0$. Then by definition of G, we have:

$$G(c_n, x, V_n) \geq G(c, x, V),$$

for all $c \in [0, x]$. Taking limits, we obtain:

$$G(c_0, x, V) \geq G(c, x, V),$$

for all $c \in [0, x]$. Hence, $c_0 = B(V)(x) = \lim_{n \to \infty} B(V_n)(x)$. Further,

$$\begin{aligned} \lim_{n \to \infty} A(V_n)(x) &= \lim_{n \to \infty} G(B(V_n)(x), x, V_n) \\ &= G(B(V)(x), x, V) = A(V)(x). \end{aligned}$$

Therefore, $\lim_{n \to \infty} T(V_n)(x) = T(V)(x)$. Since T is isotone, the iterations $T(V_n)(x)$ form an increasing sequence. Therefore, we have:

$$\sup_{n \in \mathbf{N}} T(V_n)(x) = \lim_{n \to \infty} T(V_n)(x) = T(V) = T\left(\sup_{n \in \mathbf{N}} V_n(x) \right).$$

i.e., T is monotonically-sup-preserving. Analogously, we show that T is monotonically-inf-preserving. ■

Having lemma 4.1 in hand, we are now in a position to analyze fixed points of a monotone operator T.

Theorem 4.1 (Existence of extremal SMNE) *Let assumption 1 hold. Then, the set of equilibrium values corresponding to stationary, time-consistent Markov policies is a nonempty countably chain complete poset. Moreover, there exists the greatest $\overline{h}^*$ and the least $\underline{h}^*$ time-consistent policy, and they correspond to the least $v^* = Tv^*$ and the greatest $w^* = Tw^*$ values.*

Proof: By Lemma 4.1, the operator $T : \mathcal{V} \to \mathcal{V}$ is increasing. Moreover, by Lemma 4.1, we have T is monotonically-sup-inf-preserving. As $\mathcal{V}$ is a countably chain complete poset by Theorem 3.1, T has a nonempty countably chain complete poset of fixed points, with the greatest and the least elements. ■

Theorem 4.1 is the central result on existence in our paper. First, it guarantees existence of time-consistent pure strategy equilibrium policy. Second, it asserts that the set of values has a particular poset structure, namely that it is countably chain complete poset. This in turn implies ordered sequences of time-consistent value functions have limits in this set (i.e. set of equilibrium value function are closed in the order topology on $\mathcal{V}$). Finally, for any initial state $x \in S$, there exists the greatest time-consistent value (and the least equilibrium policy), that is the optimal among time-consistent ones.

Approach used in theorem 4.1 resembles an APS/KP type of procedure for analyzing time-consistent (Bernheim, Ray, and Yeltekin (1999)) or optimal (Kydland and Prescott (1980)) sequential equilibrium strategies, but only in function spaces. In APS/KP method one constructs an operator, mapping between spaces of value correspondences (ordered by set inclusion and endowed with the week star topology), that assures self-generation property (or time-consistency in our case). Then, iterating down from the greatest element of such space of correspondences using monotone operator, one can show convergence to the greatest fixed point of this operator. This fixed point turns out to be the set of all values generated by any sequential time-consistent equilibrium. Any selection from this correspondence gives some time-consistent strategy.

In a related study (e.g., Balbus, Reffett, and Woźny (2010)), we show more generally how standard APS/KP method can be modified to a function space setting, i.e. all subsets of a space of bounded measurable functions (ordered by set inclusion, and again endowed with the weak star topology) for a broad class of stochastic games. Using such a method, and similar construction, one can obtain existence of the greatest fixed point (i.e. a set of value functions) sustainable by a sequential Markov perfect equilibrium in a large class of stochastic supermodular games and time-consistent Markov policy in hyperbolic discounting games. But if one restricts to the class of distributions assumed in 1, iterations on operator T will always track the greatest and the least equilibrium value function (and corresponding strategies) and allow to calculate the two extremal points of an APS/KP greatest fixed point set.

4.2 Computation

We next turn to the question of the computation of equilibrium. This question is particularly important in applied work (as often researchers want to simulate SMNE). We use our main existence result to prove our central theorem on the computation of extremal equilibrium values (and their supporting pure strategy SMNE). We then provide additional characterizations of equilibrium strategies that achieve these values. First, our main theorem on computation:

Theorem 4.2 (Pointwise approximation of extremal values) *Assume 1 and consider two sequences* $\{v_t\}_{t=0}^{\infty}$ *and* $\{w_t\}_{t=0}^{\infty}$ *where*[9] $v_0() = 0$, $w_0(s) = \frac{\bar{u}}{1-\delta}\mathbf{1}_{(0,\infty]}(s)$ *and* $v_t = Tv_{t-1}$ *and* $w_t = Tw_{t-1}$. *Then* $(\forall s \in S)$ $\lim_{t\to\infty} v_t(s) = v^*(s)$ *and* $\lim_{t\to\infty} w_t(s) = w^*(s)$.

Proof: Clearly $v_1 \geq v_0$. Since T is monotone, we can conclude $v_t \geq v_{t-1}$. As a result, sequence $\{v_t\}$ is increasing. As it is also bounded above, it is convergent, say to $\bar{v}$. Further, it is straightforward to show by Lebesgue Dominance Convergence theorem and Kall (1986) that $v^* = \bar{v}$. Similarly, we show that $\{w_t\}_{t=0}^{\infty}$ is decreasing and convergent to w^*. ■

Theorem 4.2 provides a simple constructive method for calculating (pointwise) two of time-consistent values, as well as their supporting policies (including those that are optimal). The theorem, though, gives us more. In particular, it allows us to calculate the pointwise bounds for any time-consistent equilibrium and as well as sustainable values. Finally, if the limits of two sequences analyzed in theorem coincide for any initial state $x \in S$, then the uniqueness of time-consistent policy is guaranteed.

In a simple example, we now show how intuitive it is to apply our methods to compute / approximate equilibrium strategies.

Example 4.1 *Consider an example of the economy with state space* $I = [0, 1]$, *utility* $u(c) = c^{\alpha}$, $g(i) = i^{\gamma}$, *while* $\lambda(y|x)$ *has a cdf given by:* y^{2-x}. *Let* $1 > \alpha > 0, 1 > \gamma > 0$.

For this economy our procedure allows to compute (by a picewise-constant approximation scheme using simple Picard iteration procedure on operator T*) the extremal MSNE. The results of our calculations*[10] *are presented in the following figures. In the first one we show convergence to the MSNE iterating both from above and below. The second figure presents comparative statics results. Sensitivity analysis shows the large discrepancies in consumption values and more importantly consumption policy slopes.*

[9] For each set A, $\mathbf{1}_A(\cdot)$ is said to be indicator of A.

[10] MATLAB program implementing our numerical procedure is available from authors upon request.

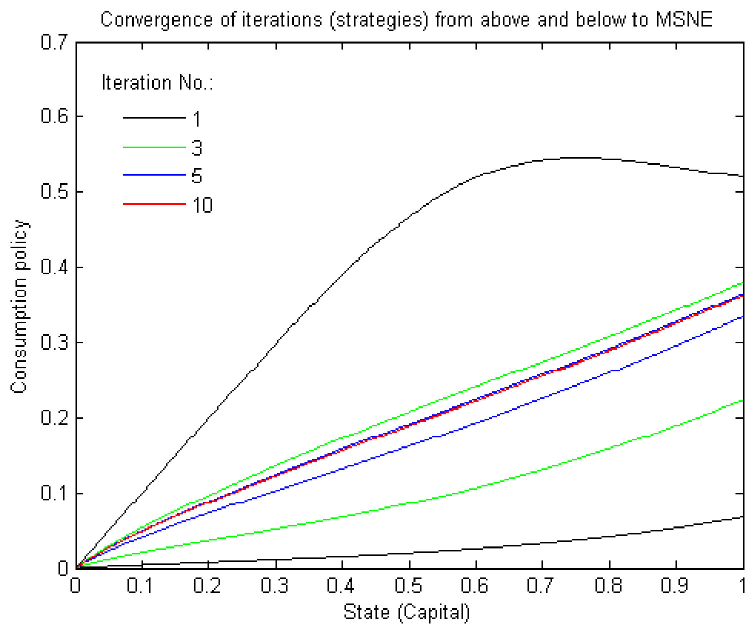

Figure 1: Convergence of iterations (policies) from above and below to MSNE ($\alpha = .3, \gamma = .5, \beta = .8, \delta = .96$).

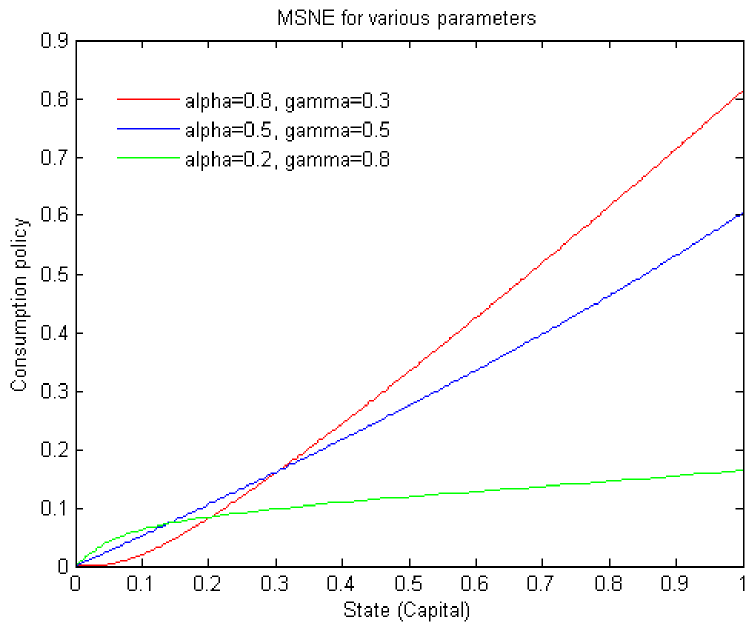

Figure 2: Consumption policy in a MSNE for $\beta = 0.9, \delta = 0.96$ and various α, γ.

The next two results allow us to further characterize (in terms of smoothness and monotonicity) the actual time consistent equilibrium policy functions h^*.

Theorem 4.3 (Monotonicity of policies) *Assume 1 and consider a time-consistent policy h^*. If $\lambda(\cdot|x)$ is constant with x, then each time-consistent equilibrium policy h^* is increasing and Lipschitz with modulus 1.*

Proof: Let $h^* = BV^*$ for some $V^* = TV^*$. Consider the function

$$G(c, x, V^*) = u(c) + \beta\delta \sum_{j=1}^{J} g_j(x - c) \int_S V^*(y)\lambda_j(dy).$$

Observe G is supermodular in c on a lattice $[0, x]$, and the feasible action set $[0, x]$ is increasing in the Veinott's strong set order. Moreover, by concavity of g_j , we conclude G has increasing differences with (c, x). By Topkis (1978) theorem argument maximizing h^* is increasing with x on S.

Similarly, if i denotes "investment", we also can rewrite this problem as:

$$H(i, x, V^*) = u(x - i) + \beta\delta \sum_{j=1}^{J} g_j(i) \int_S V^*(y)\lambda_j(dy)$$

where H is supermodular with the choice variable i on a lattice $[0, x]$, and, again, the set $[0, x]$ is increasing in the Veinott's strong set order. Again, by concavity of u, we conclude that H has increasing differences with (i, x). Therefore, again, by Topkis (1978) theorem, the optimal solution i^* is increasing with x on S.

Clearly $i^*(x) = x - h^*(x)$. Finally as both h^* and i^* are increasing on S hence h^* and i^* are Lipschitz with modulus 1.[11] ■

Notice the results in the above theorem are important, as they extend the result reported in Harris and Laibson (2001) on Lipschitz continuity of equilibrium to a broader scope of quasi-hyperbolic discount factors.

To obtain such strong characterization (both in terms of monotonicity and Lipschitz continuity) we require that λ_j are independent of state x. Although such assumption has been imposed in many related papers (see Nowak (2006) or Amir (2002)) The natural question is whether one can obtain similar results allowing for λ's dependent on x.

[11]Note, this is just a special case of a result on the existence of Lipschitz selections in Curtat (Curtat (1996), Theorem 2.3). Therefore, we could obtain a version of this result under weaker conditions on primitives involving only concavity, and not strict concavity (and using the top and bottom selections for i^*). In this case, we would guarantee the existence of Lipschitz time consistent policies (but not that each time consistent policy is Lipschitz).

Per question of Lipschitz continuity we note here that it is indeed possible. In a related work (see Balbus, Reffett, and Woźny (2009)) we propose an approach based on an implicit function theorem applied to the necessary and sufficient first order condition. We can proceed similarly here. Letting u, g_j and ζ_j be differentiable and assuming that $u'(\cdot) < 0$, while fractions $\frac{g'(\cdot)}{g''(\cdot)}$ and $\frac{\zeta_j'(\cdot)}{\zeta_j(\cdot)}$ (where $\zeta_j(x) = \int_S V^*(y)\lambda_j(dy|x)$) are bounded we can claim that equilibrium strategy h^* is Lipschitz with some constant M.

Concerning monotonicity of h^* we only mention why it may be difficult to obtain such characterization with $\lambda_j(\cdot|x)$ stochastically ordered with x. If V^* is increasing and all $\lambda_j(\cdot|x)$ are stochastically decreasing we can prove increasing differences property (between control c and state x) of W. But to assure (on a Bellman operator A) that V^* is increasing one would like to assume that $\lambda_j(\cdot|x)$ are stochastically increasing with x. Hence to get monotonicity (under this general setting) we need λ's independent on x.

We next turn to the question of continuous (but not necessarily Lipschitz continuous) time consistent policies. For this, we impose the following Feller type property on the noise.

Assumption 2 $(\forall j = 1, \ldots, J)$ $\lambda_j(\cdot|x)$ *is strongly stochastically continuous (i.e. the function* $x \to \eta_f^j(x) := \int\limits_S f(y)\lambda_j(dy|x)$ *is continuous for any* $f \in \mathcal{V}$*).*

We now prove a theorem characterizing the continuity structure of time consistent equilibrium policies.

Theorem 4.4 (Continuity of policies) *Let 1 and 2 hold. Then, each time-consistent policy equilibrium policy* h^* *is continuous.*

Proof: Let $V_{h^*} \in \mathcal{V}$ be equilibrium payoff under time-consistent policy h^*. Then, by Assumption 1, the mapping

$$x \to \zeta_{h^*}^j(x) := \int\limits_S V_{h^*}(y)\lambda_j(dy|x)$$

is continuous. Notice, the function

$$F_{h^*}(c, x) := u(c) + \beta\delta \sum_{j=1}^{J} \zeta_{h^*}^j(x) g_j(x - c)$$

is also continuous and strictly concave with respect to c for fixed $x > 0$. Let $x_n \to x_0$. Since $h^*(x) = \arg\max\limits_{c\in[0,x]} F_{h^*}(c, x)$, we have

$$F_{h^*}(h^*(x_n), x_n) \geq F_{h^*}(c, x_n).$$

Without loss of generality, suppose $h^*(x_n) \to c_0$. By the continuity of F_{h^*}, we have

$$F_{h^*}(c_0, x_0) \geq F_{h^*}(c, x_0).$$

By the strict concavity of $F_{h^*}(\cdot, x)$ and definition of h^*, we obtain $c_0 = h^*(x_0) = \lim_{n\to\infty} h^*(x_n)$. ■

4.3 Possibilities for Stationary Markov Equilibrium

We are now ready to study the computation of stationary Markovian equilibrium in our setting. In particular, we consider the question of constructing equilibrium invariant distribution generated by stochastic transition $Q(\cdot|x - h^*(x), x)$ at any time consistent policy function h^*. By Δ, we denote a family of probability measures on S. Further, define a pair of operators, namely $G : \mathcal{V} \to \mathcal{V}$, as well as $G^* : \Delta \to \Delta$ as follows:

$$G_h(f)(x) = \int_S f(y) Q(dy|x - h(x), x),$$

and

$$G_h^*(\tau)(A) = \int_S Q(A|x - h(x), x)\tau(dx).$$

The fixed points of $G_{h^*}^*$ are equilibrium invariant distributions of our economy. Before we proceed to providing conditions for the of existence and computation the set of invariant distributions, we carefully describe a case where (only) trivial invariant distribution exists.

Example 4.2 (Trivial invariant distribution) *Let $S = [0, \overline{S}]$, and take some time-consistent policy h^* with $h^*(\overline{S}) < \overline{S}$. If assumption 1, 2 hold and $\sum_{i=1}^J g_j(\overline{S}) < 1$, then the operator $G_{h^*}^*$ has a unique fixed point (corresponding to the invariant distribution δ_0).*

Proof: By theorem 4.4 h is continuous. Then,

$$\int_S f(y) Q(dy|x - h^*(x), x) = f(0)(1 - g(x - h^*(x))) + \sum_{j=1}^J g_j(x - h^*(x))\eta_f^j(x).$$

By Assumption 2, this function is stable, which implies that G_{h^*} is stable. Now define a compact operator:

$$L(f)(x) = f(0)(1 - g(x - h^*(x))).$$

We now show $G_{h^*}(f)$ is quasi-compact. Let $||f||_\infty \leq 1$.

$$|L(f)(x) - G_{h^*}(f)(x)| = \sum_{j=1}^{J} g_j(x - h^*(x)) \int_S |f(y)| \lambda_j(dy|x) \leq \sum_{j=1}^{J} g_j(x - h^*(x)).$$

By assumption, for all $x \in S$, we have $x - h^*(x) < \bar{S}$ with $h^*(\cdot)$ continuous; hence. $x - h^*(x) < \bar{S} - \epsilon$ for some $\epsilon > 0$. Therefore,

$$||L - G_{h^*}|| \leq \sum_{j=1}^{J} g_j(\bar{S} - \epsilon) < 1.$$

Hence G_{h^*} is equicontinuous. Let U be arbitrary environment of 0. Note that

$$\begin{aligned} Q(U|x - h^*(x), x) &= \\ 1 - \sum_{j=1}^{J} g_j(x - h^*(x)) + \sum_{j=1}^{J} \lambda_j(U|x) g_j(x - h^*(x)) &\geq \\ 1 - \sum_{j=1}^{J} g_j(x - h^*(x)) &\geq \\ 1 - \sum_{j=1}^{J} g_j(\bar{S} - \epsilon) &> 0. \end{aligned}$$

Hence, by a result in Futia (1982), $G^*_{h^*}$ has a unique fixed point.To see that this fixed point is δ_0, observe that by assumption 0 is an absorbing state, and each period there is a positive probability of reaching 0. ■

The above example shows conditions exist under which unique trivial invariant distribution exist, and therefore we need to proceed with care. Also, a few additional comments are in order. First, for a unique (trivial) invariant distribution to happen, the compactness of S is critical, i.e. the same argument above does not work if, for example, S is unbounded. Second, interiority of h^*, as well as the *strict* monotonicity of g play an important role. Given these, to obtain a nontrivial invariant distributions, one can take S to be unbounded from above or $\sum_{i=1}^{J} g() = 1$ for some arguments (which rules out strict monotonicity of $\sum_{i=1}^{J} g()$ on the whole domain).

Having that observation, one can easily construct situations where non-trivial invariant distributions exist.

Example 4.3 *Assume $\bar{S} < \infty$. We provide conditions for existence of a non trivial invariant distribution. Let τ be some probability distribution on $[0, \bar{S}]$. We can describe it as:*

$$\tau(\cdot) = \xi\, \tau_N(\cdot) + (1 - \xi)\delta_0(\cdot), \tag{8}$$

where τ_N is probability measure which has no atom at 0, and ξ is some constant$\in [0,1]$. If x_t has distribution τ, then the distribution of next state x_{t+1} is given by

$$\tilde{\tau}(\cdot) := G^*_{h^*}(\tau)(\cdot) = \sum_{j=1}^{J} \int\limits_S g_j^{h^*}(x)\lambda_j(\cdot|x)\tau(dx) + \int\limits_S g_0^{h^*}(x)\tau(dx)\delta_0(\cdot),$$

where $g_j^{h^*}(x) := g_j(x - h^*(x))$. Let $S_{h^*} := \{x : g_0^{h^*}(x) = 0\}$. Clearly, this is compact set. To construct invariant distribution, some additional assumptions are needed:

- *S_h is nonempty and $S_{h^*} \neq \{0\}$,*
- *for all j λ_j has Feller property and*

$$(\forall x \in S_{h^*}) \quad \sum_{j=1}^{J} g_j^{h^*}(x)\lambda_j(S_{h^*}|x) = 1. \tag{9}$$

If τ is invariant, $\tilde{\tau} = G^*_{h^*}(\tau) = \tau$. Under these assumptions, by equation (8), we have

$$\begin{aligned}
\tilde{\tau}(\cdot) := \quad & \xi \sum_{j=1}^{J} \int\limits_S g_j^{h^*}(x)\lambda_j(\cdot|x)\tau_N(dx) \\
+ \quad & \xi \int\limits_S g_0^{h^*}(x)\tau_N(dx)\delta_0(\cdot) + (1-\xi)g_0^{h^*}(0)\delta_0(\cdot) \\
= \quad & \xi \sum_{j=1}^{J} \int\limits_S g_j^{h^*}(x)\lambda_j(\cdot|x)\tau_N(dx) \\
+ \quad & \left(\xi \int\limits_S g_0^{h^*}(x)\tau_N(dx) + (1-\xi) \right) \delta_0(\cdot).
\end{aligned}$$

Since τ is invariant, and $g_0(\cdot) \geq 0$ by (8) $\int\limits_S g_0^{h^*}(x)\tau_N(dx) = 0$ unless $\xi = 0$. But if $\xi = 0$, then τ is trivial. Hence, we may assume $\xi \neq 0$. Since $g_0^{h^*} \geq 0$, τ_N must have a support in the set S_{h^*}. By (8), we have

$$\tau_N(\cdot) = \sum_{j=1}^{J} \int_{S_{h^*}} g_j^{h^*}(x)\lambda_j(\cdot|x)\tau_N(dx).$$

The last equality follows from the fact that on S_{h^*} function $g_0^{h^*} \equiv 0$.

Let us now consider a set of probability distributions with a support on S_{h^*} (say $\Delta(S_{h^*})$). Since S_{h^*} is compact, by the Prohorov Theorem (Section 5 in Billingsley (1999)), the space $\Delta(S_{h^*})$ is compact in the weak topology. On $\Delta(S_{h^*})$, define the operator

$$\mathcal{T}(\mu) := \sum_{j=1}^{J} \int_{S_{h^*}} g_j^{h^*}(x)\lambda_j(\cdot|x)\mu(dx).$$

By relation (9), we have

$$\sum_{j=1}^{J} \int_{S_{h^*}} g_j^{h^*}(x)\lambda_j(S_{h^*}|x)\tau_N(dx) = 1.$$

Hence, $\mathcal{T} : \Delta(S_{h^*}) \to \Delta(S_{h^*})$. We now show $\mathcal{T}$ has a fixed point. Since $\Delta(S_{h^*})$ is nonempty, convex and compact, we just need to show $\mathcal{T}$ is continuous in the weak topology to obtain a fixed point.. Let $\mu_n \to \mu$ weakly, and $f : S_{h^*} \to S_{h^*}$ be continuous function. Then, we have

$$\int\limits_{S_{h^*}} f(x)\mathcal{T}(\mu_n)(dx) = \sum_{j=1}^{J} \int\limits_{S_{h^*}} g_j(x) \int\limits_{S_{h^*}} f(y)\lambda_j(dy|x)\mu_n(ds).$$

By Feller properties of λ_j, we conclude that $x \to \int\limits_{S_{h^*}} f(y)\lambda_j(dy|x)$ is continuous. Hence,

$$\int\limits_{S_{h^*}} f(y)\mathcal{T}(\mu_n)(dy) \to \int\limits_{S_{h^*}} f(y)\mathcal{T}(\mu)(dy)$$

which implies $\mathcal{T}$ is continuous in the weak topology. Then, by the Schauder-Tykhonov Theorem, $\mathcal{T}$ has a fixed point τ_N^*, and the invariant distribution takes the form $\tau(\cdot) = \xi\tau_N^*(\cdot) + (1-\xi)\delta_0(\cdot)$.

In the next example, we construct another situation where the non-trivial invariant distribution is given as a convex combination of uniform distributions on a fixed interval, and the Dirac delta centered at zero.

Example 4.4 *Let $J = 1$, $\bar{S} = 5$, $\lambda(\cdot) := \mathcal{U}(2,5)$ (i.e. λ_j do not depend neither on j nor x, and has a uniform distribution on the interval $[2,5]$), $u(c) = \sqrt{c}$ and $g(i) = \min\left(\sqrt{i}, 1\right)$. Assume that β and δ satisfy: $\delta + \delta\beta\frac{14}{9} \geq 1$. First we show that for $x \in [2,5]$ $h^*(x) = x - 1$. Let $x > 2$ be an initial state. Let v_0 be a payoff under strategy $h^*(x) = x - 1$ for $x > 2$. Then*

$$v_0(x) = \sqrt{x-1} + \frac{9}{23}\int\limits_{S} v_0(y)\lambda(dy). \tag{10}$$

Since supp$(\lambda) = [2,5]$ *from (10), we have*

$$\int_S v_0(y)\lambda(dy) = \frac{14}{9} + \delta \int_S v_0(y)\lambda(dy),$$

hence

$$\int_S v_0(y)\lambda(dy) = \frac{\frac{14}{9}}{1-\delta}.$$

Since h^* *is MSNE it must solve:*

$$c \in [0,x] \to \sqrt{c} + \delta\beta \int_S v_0(y)\lambda(dy) \min\left(\sqrt{x-c}, 1\right)$$

$$= \sqrt{c} + \delta\beta \frac{\frac{14}{9}}{1-\delta} \min(\sqrt{x-c}, 1) := w(c),$$

Note that $w(c) = \sqrt{c} + \delta\beta\frac{\frac{14}{9}}{1-\delta}$ *for* $c \leq x-1$*, and* $w(c) = \sqrt{c} + \beta\delta\frac{\frac{14}{9}}{1-\delta}\sqrt{x-c}$ *otherwise. Further, right derivative of* w *in* $x-1$ *is*

$$\left.\frac{\partial w}{\partial c}\right|_{c=x-1} = \frac{1}{2\sqrt{x-1}} - \frac{1}{2}\beta\delta\frac{\frac{14}{9}}{1-\delta} \leq \frac{1}{2}\left(1 - \beta\delta\frac{\frac{14}{9}}{1-\delta}\right) \leq 0,$$

whenever $\delta + \delta\beta\frac{14}{9} \geq 1$*. This implies that* $x-1$ *is optimal policy at state* $x \in [2,5]$*. We hence show that* $\tau^*() := \xi\mathcal{U}(2,5) + (1-\xi)\delta_0$ *is invariant distribution under strategy* h^* *for arbitrary* $\xi \in [2,5]$*. Indeed if* $\tau_t =^d \tau$ *then*[12]*:*

$$\tau_{t+1} =^d \xi\lambda + (1-\xi)\delta_0 =^d \tau_t.$$

4.4 Monotone Comparative Statics

Finally, motivated by the indeterminacy result in Gong, Smith, and Zou (2007) (as well as concerns about the possible econometric estimation of our stochastic game), we now consider a parameterized version of our optimization problem in the previous section of the paper. For a partially ordered set Θ , with $\theta \in \Theta$ a typical element, define the greatest and least time-consistent policies as $\overline{h}^*_\theta$ and $\underline{h}^*_\theta$, respectively.

We make the following assumption.

Assumption 3 *Let us assume:*

- $u : S \times \Theta \to \mathbb{R}$, $c \to u(c,\theta)$ *is continuous, increasing and strictly concave on* S *with* $(\forall \theta \in \Theta)\, u(0,\theta) = 0$*. Also* u *has increasing differences with* (c,θ) *and* $\theta \to u(c,\theta)$ *is decreasing.*

[12] $X =^d Y =^d \tau$ means that random variable X has the same distribution as Y, and it is τ.

- *For any $x, i \in S$ and $\theta \in \Theta$ let $Q(\cdot|i, x, \theta) = (1 - g(i, \theta))\delta_0(\cdot) + \sum_{j=1}^{J} g_j(i, \theta)\lambda_j(\cdot|\theta)$.*
- $(\forall j = 1, \ldots, j)\, g_j : S \times \Theta \to [0, 1]$ *and $i \to g_j(i, \theta)$ is continuous, increasing and concave with $(\forall \theta \in \Theta)\, g_j(0, \theta) = 0$. Also g_j has decreasing differences with (i, θ) and $\theta \to g_j(i, \theta)$ is decreasing on Θ.*
- *δ_0 is a delta Dirac measure concentrated at point 0, while $(\forall j = 1, \ldots, j)\, \lambda_j(\cdot|\theta)$ is a Borel transition distribution on S for any $\theta \in \Theta$, where $\lambda_j(\cdot|\theta)$ is stochastically decreasing with θ.*

We Assumption 3 in place, we can now prove our main result on monotone comparative statics for extremal time consistent equilibrium policies.

Theorem 4.5 (Monotone comparative statics) *Let Assumption 3 be satisfied. Then, the mappings $\theta \to \overline{h}^*_\theta$ and $\theta \to \underline{h}^*_\theta$ are both increasing on Θ.*

Proof: By theorem 4.1, for any $\theta \in \Theta$, there exist top and bottom time consistent policies $\overline{h}^*_\theta$ and $\underline{h}^*_\theta$. By theorem 4.3, $\overline{h}^*_\theta(x)$ and $\underline{h}^*_\theta(x)$ (as well as $x - \overline{h}^*_\theta(x)$ and $x - \underline{h}^*_\theta(x)$) are increasing functions of $x \in S$. As a result, for each θ the operator T_θ maps $\mathcal{V}$ into increasing functions, hence, the fixed points of T_θ are increasing function of $x \in S$.

Now, for increasing $V \in \mathcal{V}$, consider a function

$$G(c, x, \theta, V) = u(c, \theta) + \beta\delta \sum_{j=1}^{J} g_j(x - c, \theta) \int_S V(y)\lambda_j(dy|\theta),$$

and observe that G is decreasing with θ, and has increasing differences with (c, θ). Clearly, $A_\theta V(x) = \max_{c \in [0,x]} G(c, x, \theta, V)$ is decreasing with θ. Similarly, by Topkis (1978) theorem, $B_\theta V(x)$ is increasing with θ (where $B_\theta V(x) = \arg\max_{c \in [0,x]} G(c, x, \theta, V)$). Consequently, we have

$$\theta \to T_\theta V(x) = \frac{1}{\beta} A_\theta V(x) - \frac{1 - \beta}{\beta} u(B_\theta V(x)),$$

is decreasing on Θ. From theorem 3.1, we therefore conclude the greatest w^*_θ and the least fixed point v^*_θ are decreasing with θ. Consequently, $\theta \to G(c, x, \theta, w^*_\theta)$ is decreasing and $(c, \theta) \to G(c, x, \theta, w^*_\theta)$ has increasing differences with (c, θ). Then, by Topkis (1978) theorem, $\underline{h}^*_\theta$ is increasing with θ. The reasoning is similar for v^*_θ. ■

5 Conclusion

Their are numerous reasons why developing constructive methods in the study of (sequential or Markov perfect) equilibrium in models with dynamic strategic interaction can be important. The first reason steps from the increasingly important role that numerical methods have played in the characterization of dynamic equilibrium over the last few decades (e.g., given the recent interest in many literatures that seek to fit the equilibrium of these models to the data via either calibration or estimation techniques). Indeed, one of the important contributions that the generalized Euler equation methods (of Harris and Laibson (2001)) have opened, is the possibility of developing a large catalog of tractable methods for characterizing the structure of Markov perfect equilibrium in dynamic games. In this spirit, and in the context of a stochastic game representation of the problem of modeling consumers with hyperbolic preferences, we are able to make significant advances on these types of methods. In particular, our methods allow us to compute *both* time-consistent optimal policies and associated value functions, with neither of these equilibrium objects being computed using the local approximation arguments implicitly given in a generalized Euler equation method of Harris and Laibson (2001). Therefore, in an important sense, our work can be viewed as a direct extension of theirs.

Also, more generally, our methods show that stochastic games can provide an very useful setting for studying dynamic problems with strategic interactions between "generations" of agents. In such problems, when constructive methods are available, these methods prove useful also for understanding not only issues related to the approximation/estimation of equilibrium (in a parameter), but they also provide the possibility of exploring numerous important theoretical questions that are not so readily addressed with pure topological methods (e.g., questions concerning equilibrium comparative statics, and understanding the theoretical differences between various models under consideration).

Per this latter theoretical issue, for example, our Theorem 4.1 asserts existence of the greatest value and hence the optimal time-consistent policy. That is, among Markov equilibria of our game, we show there exists an optimal, time-consistent policy which can be interpreted inducing a infinite sequence of policies stemmed from one undertaken by a "planner", where under a continuation of the state variable, this sequence of future planned policies would actually be implemented by the continuation "doers" in the subsequent periods. Such a situation has a natural interpretation as a "sustainable" policy for a decision maker with hyperbolic preference, and this idea can be integrated nicely into models of optimal policy design with global pollution (e.g., see the work of Karp and Tsur (2008)). Actually, our results on the structure of optimal time-consistent consumption decisions have direct implications for the recent work on welfare analysis and public policy in general (e.g., see the recent work of Nakajima (2010)).

Finally, we can understand the relationship between our work and that of recent authors working in decision theory as follows: consider the following decision problem:

$$\max_{\{c_t\}_{t=0}^{\infty}} u(c_0) + E_t \sum_{t=1}^{\infty} \delta^t u(c_t),$$
$$\text{s.t. } \forall t \, \{c_i\}_{i=t}^{\infty} \text{ solves } u(c_i) + \beta E_t \sum_{i=t+1}^{\infty} \delta^{i-t} u(c_i).$$

If we restrict attention to stationary policies, we have

$$\max_{h \in \mathcal{H}} V_h(s_0),$$
$$\text{s.t. } h(x) \in \arg\max_{c \in [0,x]} \left\{ u(c) + \beta\delta \int_S V_h(y) Q(dy|x-c, x) \right\}. \quad (11)$$

From Theorem 4.1, we know (i) the solution to such maximum problem exists, and (ii) this solution is the least element (for any initial state s_0) from the set of all Markov equilibria for our stochastic game. Therefore, Problem (11) can be interpreted as being similar to the problem studied in Gul and Pesendorfer (2001), where the authors ask the question of how decision makers overcome temptation (i.e., understanding a Strotz representation for optimal choice under temptation driven by preferences). Here, every period a decision is taken by a tempted "doer" with ties are resolved in favor of the "planner". In this context, a natural question that arises is whether our stochastic game methods can be used to "represent" either the GP or DLR preferences. We can use our results in the paper to discuss the possibilities, but defer more rigorous answers to these issues for future work. First, Benabou and Pycia (2002) have proposed a game representing outcomes of the Gul and Pesendorfer (2001) preference representation result for such models with temptation and self-control. Specifically, Benabou and Pycia (2002) consider a self-control game between two agents (planner with utility $u + v$, and a doer with utility v) choosing a costly input that increases their chances of winning a game and choosing an item from a menu. Evaluating the expected outcome in the Nash equilibrium of this game, under preferences u, one obtains a GP preference representation. Here, we mention that the Benabou and Pycia (2002) game can be easily extended using our results to a dynamic, recursive setting, where "long term" preferences of the decision maker are given by: $u + \delta U$, while the "short term" preferences are given by v (or, if we consider dynamic temptation, we let this be given by $v + \delta V$). Here, the objects U and V in the agents dynamic preferences are continuation value functions determined in the equilibrium of the stochastic game. In this sense, we could think of the Gul and Pesendorfer (2004) or Noor (2007a,b) dynamic self-control representations as resulting from a stochastic game using constructive techniques developed in this paper. Of course, the problem with the dynamic Benabou and Pycia (2002) game is that the value function U must correspond to the *long-run* preferences u; hence, it is the value of neither player playing the game of self-control. Therefore, although essentially we can represent the Gul and Pesendorfer (2004) dynamic preferences using a game theoretic interpretation, it becomes a sequence of independent, static games.

Second, Fudenberg and Levine (2006) study a game between planner and a sequence of myopic doers. In this game, the doers choose actions, while the planner choose costs. They show that in a subgame-perfect equilibria of such a game, the solution is equivalent to a maximization problem of a planner (choosing sequence of costs) under incentive compatibility constraints imposed by the actions preferred by the doers Fudenberg and Levine (2006) also discuss how the payoff and costs functions should be specified for their game (or equivalent planner problem) to yield the GP preference representation (under additional assumptions). We can, again, use our stochastic game/or decision problem for a study of Fudenberg and Levine (2006) representation of GP decision problem constructively.

Finally, Dekel and Lipman (2010) introduce random GP and random Strotz representation of preferences. They show that random Strotz agent can represent any GP and random GP preference structure. The converse is true, but only in the case of Lipschitz continuous preferences. Again, by adding noise to their Strotz decision problem Dekel and Lipman (2010) brings self-control literature close to our stochastic game approach.

The above mentioned points suggests that techniques proposed in this paper have wide applications, that can be used in future work on self-control.

A Appendix: Proof of technical result

Let $(X, \leq)$ be a countably chain complete poset, i.e. where each increasing sequence has supremum, and each decreasing sequence has infimum. Assume that X has the greatest element $\overline{\theta}$ and the least element $\underline{\theta}$. For a monotone sequence $\{x_n\}_{n=0}^{\infty}$, let

$$\bigvee x_n := \sup_{n \in \mathbf{N}} x_n,$$

and

$$\bigwedge x_n := \inf_{n \in \mathbf{N}} x_n.$$

Denote by $F^n(x)$ the n-th orbit (or iteration) of x under the function F, i.e. $F^n(x) = F \circ F \circ \ldots \circ F(x)$. We have the following two theorems, with the first pertaining to countably chain complete posets, and the second pertaining to $\sigma-$complete lattices.

Theorem A.1 *Let X be a countably chain complete poset and $F : X \to X$ an increasing function, that is monotonically sup-inf-preserving i.e.*

- *if x_n is increasing, then $F\left(\bigvee x_n\right) = \bigvee F(x_n)$ and*
- *if x_n is decreasing, then $F\left(\bigwedge x_n\right) = \bigwedge F(x_n)$.*

Then:

(i) $\overline{\Phi} := \bigwedge F^n(\overline{\theta})$ is the greatest fixed point and $\underline{\Phi} := \bigvee F^n(\underline{\theta})$ is the least fixed point.

(ii) the set of fixed points is a nonempty countably chain complete poset with

$$\overline{\Phi} = \bigvee \{x : F(x) \geq x\}, \tag{12}$$

and

$$\underline{\Phi} = \bigwedge \{x : F(x) \leq x\}, \tag{13}$$

Proof: Proof of (i): Clearly $F(\overline{\theta}) \leq \overline{\theta}$. If for some n, $F^n(\overline{\theta}) \geq F^{n+1}(\overline{\theta})$, then $F^{n+1}(\overline{\theta}) = F(F^n(\overline{\theta})) \geq F(F^{n+1}(\overline{\theta})) = F^{n+2}(\overline{\theta})$. Hence, $F^n(\overline{\theta})$ is decreasing, and $\overline{\Phi}$ is well defined. Since F is monotonically inf-preserving, we have

$$\begin{aligned} F(\overline{\Phi}) &= F\left(\bigwedge F^n(\theta)\right) \\ &= \bigwedge F^{n+1}(\theta) \\ &= \overline{\Phi} \end{aligned}$$

Therefore, $\overline{\Phi}$ is fixed point of F. Let us take arbitrary fixed point $e = F(e)$. Clearly, $e \leq \overline{\theta}$, and $e = F(e) \leq F(\overline{\theta})$. If $e =\leq F^n(\overline{\theta})$, then $e = F(e) =\leq F^{n+1}(\overline{\theta})$. Therefore, $e \leq F^n(\overline{\theta})$ for all n, which implies $e \leq \overline{\Phi}$. Similarly, we prove that $\underline{\Phi}$ is well defined and it is the least fixed point of F.

Proof of (ii): Let e_n be an increasing set of fixed points. Let $\bar{e} = \bigvee e_n$. Then,

$$\begin{aligned} F(\bar{e}) &= F\left(\bigvee e_n\right) \\ &= \bigvee F(e_n) \\ &= \bigvee e_n = \bar{e} \end{aligned}$$

Similarly, we prove the thesis for decreasing sequences. Now, we finally prove equality (12). Let x be arbitrary point such that $x \leq F(x)$. Clearly $x \leq \overline{\theta}$. Assume $x \leq F^n(\overline{\theta})$. Then, $x \leq F(x) \leq F(F^n(\overline{\theta})) = F^{n+1}(\overline{\theta})$. Hence, $x \leq \overline{\Phi}$. Since $\overline{\Phi} \in \{x : F(x) \geq x\}$, equality (12) is proven. We prove (13) analogously. ■

Theorem A.2 *Let X be a σ-complete lattice with the greatest and the least elements. Let $F : X \to X$ be increasing and sup-inf preserving function. Then:*

(i) $\overline{\Phi} := \bigwedge F^n(\overline{\theta})$ is the greatest fixed point and $\underline{\Phi} := \bigvee F^n(\underline{\theta})$ is the least.

(ii) the set of fixed point is a nonempty σ-complete lattice and

$$\overline{\Phi} := \bigvee \{x : F(x) \geq x\}, \tag{14}$$

and

$$\underline{\Phi} := \bigwedge \{x : F(x) \leq x\}. \tag{15}$$

Proof: To show (i), since any σ-complete lattice is a countably chain complete poset, statement (i) follows from Theorem A.1. Analogously as in Theorem A.1, we obtain (15) and (14). To show (ii) (i.e., the fixed point set forms a σ-completely lattice), we take sequence of fixed points e_n and supremum $\bar{e} := \bigvee e_n$. By sup-inf preserving property of F, we obtain:

$$\begin{aligned} F(\bar{e}) &= F\left(\bigvee e_n\right) \\ &= \bigvee F(e_n) \\ &= \bigvee e_n = \bar{e}. \end{aligned}$$

Analogously, we could show that $\underline{e} := \bigwedge e_n$ is fixed point of F. ■

We finally prove a theorem (and a corollary) on increasing selections for parameterized problems that we use in the paper to obtain our results on equilibrium monotone comparatives.

Theorem A.3 *Let X be a countably chain complete poset with the greatest and least elements and T a poset. If $F : X \times T \to X$ is increasing, and monotonically-sup-inf preserving on X then $t \to \overline{\Phi}(t)$ and $t \to \underline{\Phi}(t)$ are isotone.*

Proof: Let $t_1 \leq t_2$. From Theorem A.1 we know that $m_i := \overline{\Phi}(t_i) = \vee\Gamma_i := \vee\{x : F(x, t_i) \leq x\}$. Note that by isotonicity of $F(x, \cdot)$ we obtain $m_1 = F(m_1, t_1) \leq F(m_1, t_2)$. Hence $m_1 \in \Gamma_2$. Since m_2 is the greatest element of Γ_2, hence $m_1 \leq m_2$. ■

Corollary A.1 *Let X a σ-complete lattice with the greatest and the least elements and T a poset. Let F be increasing, and sup-inf preserving on X. Then $t \to \overline{\Phi}(t)$ and $t \to \underline{\Phi}(t)$ are isotone.*

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