

Equilibria in Large Games with Strategic Complementarities*

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Abstract

We study a class of static games with a continuum of players and complementarities. Using monotone operators on the space of distributions, we prove existence of the greatest and least distributional Nash equilibrium under different set of assumptions than one stemming from Mas-Colell (1984) original work, via constructive methods. In addition, we provide computable monotone comparative statics results for ordered perturbations of the space of our games. We complement our paper with results concerning equilibria on strategies as introduced by Schmeidler (1973). Finally we discuss the equilibrium uniqueness and present applications for Bertrand competition, general equilibrium models, and so called "beauty contests".

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1 Introduction and related literature

Since the seminal papers of Schmeidler (1973) and Mas-Colell (1984) game theory literature has analyzed so called large games, i.e. games played by a measure space of players. Despite obvious similarities (see discussion in (Khan, 1989) a.o.), approaches undertaken by both authors have significant differences. Schmeidler (1973) studies a game where players' payoffs depend on their own actions and action profile of all other players. That is, in his game it matters for a player's payoff, who (of his/her opponents) takes each action. Hence, Schmeidler defines a Nash equilibrium of the large game in *strategies*, i.e. functions from the set of players to their action sets. This is different from Mas-Colell (1984) approach, who studies games where players' payoffs depend on their own action and the *distribution* on all players actions. Consequently he defines a Nash equilibrium in *distributions* on both: players' characteristics and actions. As a result, a term *anonymous* game is justified, as it does not matter who chooses each action. Here let us mention that anonymity can be also modeled using Schmeidler (1973) approach, where each player's payoff depends on his/her own action and *average* of players strategies. Despite these differences both approaches seem natural generalizations of Nash equilibria of small games (i.e. games played by finite number of players) and proved useful in studying economic problems where influence of a particular player on the equilibrium aggregates is insignificant (see Horst and Scheinkman (2009)).

The central question in the streams of literature starting from both papers concerned conditions for equilibrium existence. In their seminal papers both authors (Mas-Colell and Schmeidler) use topological fixed points theorems of Fan-Glicksberg applied on a continuous best response

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map. Since these results a vast literature has studied possibilities of generalizing existence results for this class of games. Specifically equilibrium existence results (for both types of equilibria) have been generalized (i) by Khan (1986, 1989), Khan, Rath, and Sun (1997) or Balder (1999) allowing for general spaces of players' actions; (ii) by Rath (1996) to the case of uppersemicontinuous payoffs; (iii) by Balder and Rustichini (1994) and Kim and Yannelis (1997) to Bayesian games¹; and (iv) by Martins da Rocha and Topuzu (2008) to non-ordered preferences. For a survey of more recent results we refer² the reader to a chapter by Khan and Sun (2003).

The second strand of game theory literature, we discuss here, analyze so called supermodular games or games with strategic complementarities (see Topkis (1979), Milgrom and Roberts (1990), Veinott (1992), Zhou (1994) and more recently Heikkilä and Reffett (2006)). In such games the best response maps are increasing (but not necessarily continuous) due to complementarities in the payoff function between own and other strategies. Then applying the Tarski (1955) or Veinott (1992)/Zhou (1994) fixed point theorem on a complete lattice (i.e. a partially ordered set in which any subset has a supremum and infimum) of the (product) action space, one concludes existence of complete lattice of Nash equilibria. For applications of such games in economic theory we refer the reader to Topkis (1998) book.

In this paper we study equilibria of static games with a continuum of players using monotone operators on the space of distributions. Hence we link the large games literature with that on (quasi)-supermodular games. The questions we address concern equilibrium existence, their computation and equilibrium comparative statics. Specifically, using fixed point theorem of Markowski, we prove existence of a distributional Nash equilibrium under different set of assumptions than Mas-Colell (1984) via constructive methods. In addition, we provide computable monotone comparative statics results for ordered perturbations of the space of our games. Similarly, using the same theorem we complement our paper with results concerning (pure strategy) equilibria on strategies as introduced by Schmeidler (1973). As our methods are constructive we are able to develop techniques for equilibrium computation that cannot be addressed using topological results directly. This significant difference is our main motivation for using order-theoretical tools. Finally we present conditions for existence of symmetric equilibrium and equilibrium uniqueness.

Although tools used for a study of small and large games with complementarities are similar the results are not. Specifically, we show conditions under which the set of distributional equilibria has the greatest and the least elements but is not a complete lattice. Similarly for the set of Nash equilibria of a Schmeidler game. Moreover if best response is a function then the set of distributional (resp. Schmeidler) equilibria is a (resp. countably) chain complete poset, i.e. partially order set of in which any ordered, (resp. countable) subset has supremum and infimum). The differences between small and large games, come from the infinite dimensionality of the large one, the feature that is absent in their small counterparts. Moreover, in the literature we cited above, technical properties of large games played a role, in their applicability and economists' interest. Specifically convexifying effects of aggregation were used to dispense with the convexity assumptions on both action sets and preferences. We should mention that it is not a case of our paper, as we use order theoretical and not topological techniques. For these reasons we think that it is of interest for applied economists to analyze a class of large (quasi)-supermodular games.

Finally we mention that there is a related body of work on economies with a continuum of agents including papers stemming from Aumann (1975) contribution. As the non-cooperative game theory tools can be used for a study of general equilibrium problems (Arrow and Debreu, 1954), we state an example stressing applicability of our tools in such class of problems (see also discussion in Hart, Hildenbrand, and Kohlberg (1974), Khan (1985) and Balder (2008)). Apart from that we also present applications of our results for Bertrand competition and beauty

¹Compare also with Balder (2002) unifying approach to equilibrium existence.

²We also report Blonski (2005) results on equilibrium distributions characterization and Rashid (1983) results on approximation of Nash equilibria by equilibria in games with finite number of players.

contests.

The rest of the paper is organized as follows. In section 2 we prove distributional equilibrium existence and equilibrium comparative statics. In section 3 we prove similar result for Nash equilibria of large games as introduced by Schmeidler (1973). Then (section 4) we discuss conditions for symmetric equilibrium uniqueness. The economic applications are analyzed in section 5. Finally section 6 concludes, while in section 7 we state definitions and auxiliary fixed point results used in the paper as well as proofs of all theorems.

2 Distributional equilibria in large games

Let Λ be a poset of players. Endow Λ , with σ -field \mathcal{L} and nonatomic, σ -finite measure λ on it. Mathematically $(\Lambda, \mathcal{L}, \lambda)$ is a measure space, where Λ it is assumed to be compact metrizable space w.r.t. corresponding *order topology*, and \mathcal{L} is a σ -field of open sets. Let A be an action set, while $\tilde{A} : \Lambda \rightarrow 2^A$ is a compact valued and *weakly measurable* correspondence assigning for any player its subset of $A \subset \mathbb{R}^m$ $m \in \mathbb{N}$ (endowed with natural product order and Euclidian topology). Let \mathcal{A} be a family of Borel subsets on A . Let $\mathcal{M}(\Lambda \times A)$ be a set of probability distributions on $\mathcal{L} \otimes \mathcal{A}$, with marginal distribution λ on Λ .

Endow $\mathcal{M}(\Lambda \times A)$ with a *first order stochastic dominance* order $\succeq_{\mathcal{P}}$, where $\Lambda \times A$ has a product order. Endow also $\mathcal{M}(\Lambda \times A)$ with a *weak topology* \mathcal{P} : $\tau_n \rightarrow_{\mathcal{P}} \tau$ (or $\lim_{n \rightarrow \infty} \tau_n = \tau$ in \mathcal{P})
Let

$$\mathcal{D} := \{\tau \in \mathcal{M}(\Lambda \times A) : \tau(Gr(\tilde{A})) = 1 \text{ for a.e. } \alpha \in \Lambda\}$$

be a set of distributions, where $Gr(\cdot)$ is a graph of the correspondence. A player $\alpha \in \Lambda$ is characterized by a payoff function $r : \Lambda \times A \times \mathcal{D} \rightarrow \mathbb{R}$. We summarize a game by $\Gamma := ((\Lambda, \mathcal{L}, \lambda), A, \tilde{A}, r)$.

Definition 2.1 (Distributional equilibrium, Mas-Colell (1984)) *A distributional (Nash) equilibrium of Γ is a measure τ^* on $\Lambda \times A$ such that marginal distribution τ^* on Λ is λ and*

$$\tau^*(\{(\alpha, a) : r(\alpha, a, \tau^*) \geq r(\alpha, a', \tau^*) \forall a' \in \tilde{A}(\alpha)\}) = 1.$$

Our aim is to prove existence of a distributional equilibrium. For this reason we need some assumptions.

Assumption 2.1 *Assume that:*

- *correspondence \tilde{A} is compact valued, complete lattice valued and weakly λ -measurable,*
- *for λ -almost every player α function r is quasisupermodular in a , while r satisfy single crossing property in (a, τ) ,*
- *for any $\tau \in \mathcal{D}$, $(\alpha, a) \rightarrow r(\alpha, a, \tau)$ is a Caratheodory function, i.e. is continuous with a and measurable in α .*

Our proof of existence of distributional equilibria involves constructing the best response operator defined on a partially ordered space and then application of Markowski fixed point theorem (see technical appendix, theorem 7.1).

First we construct a useful operator. Define a correspondence

$$m(\alpha, \tau) := \arg \max_{a \in \tilde{A}(\alpha)} r(\alpha, a, \tau)$$

and the functions $\bar{m}(\alpha, s) := \bigvee m(\alpha, \tau)$ and $\underline{m}(\alpha, s) := \bigwedge m(\alpha, \tau)$ (in the whole paper \bigvee (\bigwedge) returns the greatest (least) element). Observe that each distributional Nash equilibrium satisfies $\tau^*(\{(\alpha, a) : a \in m(\alpha, \tau^*)\}) = 1$. We have

$$\bar{T}_\alpha(\tau)(\cdot) = \delta_{\bar{m}(\alpha, \tau)}(\cdot),$$

where for each $\alpha \in \Lambda$ and $\tau \in \mathcal{D}$, $\delta_{\bar{m}(\alpha, \tau)}$ denotes a distribution on A which is equal Dirac delta concentrated at $\bar{m}(\alpha, \tau)$. Analogously, we define \underline{T}_α , concentrated in $\underline{m}(\alpha, s)$. Let $\bar{T} : \mathcal{D} \rightarrow \mathcal{D}$ and for all $G \in \Lambda \otimes A$

$$\bar{T}(\tau)(G) := \lambda\{\alpha : (\alpha, \bar{m}(\alpha, \tau)) \in G\}.$$

Define \underline{T} analogously.

Lemma 2.1 *Operators \bar{T} and \underline{T} are well defined. Moreover, \bar{T} and \underline{T} are increasing w.r.t. \succeq_P order.*

In Hopenhayn and Prescott (1992) in Proposition 1 it is proven that every set of probability distributions forms a chain complete poset w.r.t. first order stochastic dominance order. The result is based on Kamae, Krengel, and O'Brien (1977) theorem. The lemma below shows also, that a limit of the sequence coincide with first order stochastic dominance supremum for countable subsets.

Lemma 2.2 (Hopenhayn and Prescott (1992)) *(\mathcal{D}, \succeq_P) is a chain complete poset. Moreover, for any \succeq_P increasing (decreasing) sequence τ_n $n \in \mathbb{N}$*

$$\bigvee_{n \in \mathbb{N}} \left(\bigwedge_{n \in \mathbb{N}} \right) \tau_n = \lim_{n \rightarrow \infty} \tau_n.$$

Let $\bar{\delta}, \underline{\delta}$ denote the greatest and least element of \mathcal{D} . We are now ready to establish our main result.

Theorem 2.1 *Under assumption 2.1*

- *There exist the greatest ($\bar{\tau}^*$) and the least ($\underline{\tau}^*$), w.r.t. \succeq_P order, distributional equilibrium of game Γ .*
- *If $\underline{T} = \bar{T}$ then the set of distributional equilibria is a chain complete poset.*
- *Assume additionally that r is continuous in (a, τ) w.r.t. product of Euclidian topology and topology \mathcal{P} . Then we have: $\bar{\tau}^* = \lim_{n \rightarrow \infty} \bar{T}^n(\bar{\delta})$ and $\underline{\tau}^* = \lim_{n \rightarrow \infty} \underline{T}^n(\underline{\delta})$.*

Remark 2.1 *We can relax assumptions of theorem 2.1 to prove existence of the greatest (resp. least) distributional equilibrium of game Γ . Specifically we can replace quasi-supermodularity with r being join (resp. meet) subextremal, while single crossing property with join (resp. meet) upcrossing differences. Similarly we can relax assumption that A is complete lattice valued to the case of complete join (resp. meet) lattice valued. See Calzi and Veinott (1992) for the details.*

We now compare our results to the others concerning distributional equilibria in large games and small games with strategic complementarities. Apart from equilibrium existence, we think that all the other results for distributional equilibria for large games are new.

Firstly, the theorem above establishes existence of distributional equilibria under different assumptions than Mas-Colell (1984) and following related papers. On one hand we manage to dispense the continuity assumption³ of r with τ . (see also discussion in Rath (1996) concerning uppersemicontinuous payoffs). Instead we define an order on Λ and D , and add quasisupermodularity, single crossing property of r and complete lattice structure of the action set. As a result, in the class of games with strategic complementarities, our result offers a tool for analyzing games with payoff discontinuities⁴

³The second part of our theorem assumes, however, payoff continuity but convergence is defined in an order-topology and needed for equilibrium approximation only.

⁴Consider an example of a large Bertrand game where profits may be discontinuous due to products substitutability. Observe that this cannot be modeled under Rath (1996) assumptions.

Moreover, the theorem above establishes existence of the greatest and the least distributional equilibria. This result is counterparting existence theorem for small (quasi-)supermodular games (see Veinott (1992) and Zhou (1994)). Unlike small (quasi-)supermodular games, large games do not necessarily have a complete lattice of distributional equilibria. The reason is that, although action set A is a complete lattice, the set of distribution on A need not be a complete lattice. The example is the set of distributions on $A \subset \mathbb{R}^m (m \geq 2)$ ordered by first order stochastic dominance (see Kamae, Krengel, and O'Brien, 1977). In such a case we are only able to show that the set of distributions has greatest and least elements and for single valued best-response forms a chain complete poset. As a result, instead of Tarski fixed point theorem, we need to use our generalization of Markowski fixed point theorem.

Second, our monotone methods approach directly suggest ways of distributional equilibrium computation (compare Topkis, 1998, chapter 4.3). To weekly compute the greatest equilibrium one need to calculate the (week) limit of the sequence of distributions generated on \bar{T} , iterated down from the greatest element of the set of distributions. It is possible since under additional continuity assumption \bar{T} and \underline{T} are also monotonically-inf/sup-preserving. Observe that in the original proof of Mas-Colell (1984) and later related results the continuity of each payoff with distribution τ is critical for showing existence. In our case we do not need that assumption for existence but under this assumption establish equilibrium approximation.

Third, we note that our restrictive assumption on action set (being compact and complete lattice in \mathbb{R}^m), are needed to guarantee that best response map possesses the greatest and the least elements and are order-continuous. We leave the question, whether it is possible to generalize our results to the class of Banach action spaces, to further work (compare with Khan (1989)).

Fourthly, one can use our generalization of Markowski theorem and establish equilibrium comparative statics. For this reason consider a measure space of parameters (S, \mathcal{S}, μ_s) . Endow S with some partial order denoted \leq_S . Let us take a static game $\tilde{\Gamma}$ depending on parameter $s \in S$. More precisely let $\Gamma(s) = ((\Lambda, \mathcal{A}, \lambda), A(\cdot, s), \tilde{A}(\cdot, s), r(\cdot, s))$. Assume for $\Gamma(s)$:

Assumption 2.2 *For all $s \in S$ correspondence $\tilde{A}(\cdot, s)$ and $r(\cdot, s)$ satisfy Assumption 2.1. Moreover, assume $s \Rightarrow \tilde{A}(\alpha, \tau, s)$ is Veinott strong set order increasing. For λ -almost every player α function r satisfy single crossing property in (a, s) . Finally assume that $(\alpha, s) \rightarrow \max_{a \in \tilde{A}(\alpha, s)} r(\alpha, \tau, a, s)$ is $\mathcal{L} \otimes \mathcal{S}$ measurable.*

Under this assumption we can prove the next equilibrium comparative statics theorem that finishes this section.

Theorem 2.2 *Assume 2.2. Then the greatest and the least distributional equilibria of $\Gamma(s)$ are pure. Moreover, both are $\mathcal{L} \otimes \mathcal{S}$ measurable as a functions of (α, s) and isotone on S .*

The only result concerning equilibrium comparative statics for large games, that we are aware of, was recently offered by Acemoglu and Jensen (2010). Their approach is very similar to ours, as the best response map in their game has increasing (with parameter) selections (see their definition 3). The class of games they analyze, however, is different from ours as they concentrate on aggregative games, where the aggregate is given by an integral of actions of others. Hence our class of games is more general. For single dimensional action space A Acemoglu and Jensen (2010) manage to show comparative statics of the extremal (aggregative) equilibria using results of Milgrom and Roberts (1994) without SCP between player actions and aggregates. This is more general than our result applied to their game directly. However, for multidimensional aggregates case, they require increasing differences in the action of any player and the aggregate, while we require SCP only. Finally Acemoglu and Jensen (2010) use topological fixed point theorem of Kakutani to show equilibrium (aggregate) existence. On the contrary, we use the order-theoretical fixed point results that allow for equilibrium computation.

3 Nash equilibria in nonatomic games

In this section we consider Nash equilibria of Schmeidler large games. We start with the parameterized game.

The space of players is $(\Lambda, \mathcal{L}, \lambda)$, and satisfies the same assumptions as in section 2. Let (S, \mathcal{S}, μ_S) be a measure space, where S is a poset with some partial order. Assume that $\tilde{A}(\alpha, s) \subset \mathbb{R}^m$ ($m \in \mathbb{N}$) is compact, and a complete lattice w.r.t. standard product order. Let $M(\mathcal{L} \otimes \mathcal{S})$ be a set of measurable functions from $\Lambda \times S$ to \mathbb{R}^m with respect to product σ -field. Let

$$\tilde{D} := \left\{ f : \Lambda \times S \rightarrow \mathbb{R}^m : f(\alpha, s) \in \tilde{A}(\alpha, s) \right\},$$

and $D := \tilde{D} \cap M(\mathcal{L} \otimes \mathcal{S})$. By Ryll-Nardzewski Theorem D is nonempty. Equip D and \tilde{D} with a standard product order and standard product topology. The payoff function is $r : \Lambda \times A \times D \times S \rightarrow \mathbb{R}$. A strategy profile is said to be an element of D . The current payoff for player α from strategy profile $f \in D$, is $r(\alpha, f(\alpha, s), f(\cdot, s), s)$. Such a game is called a nonatomic game and denoted by $\Gamma = ((\Lambda, \mathcal{L}, \lambda), A, \tilde{A}, r, S)$.

Now let $s \in S$ be given and assume that players use strategy $f \in D$. Under semicontinuity of r with respect to a we can define a correspondence $BR : D \rightrightarrows \tilde{D}$ such that:

$$BR(f)(\alpha, s) := \arg \max_{a \in A(\alpha, S)} r(\alpha, a, f, s).$$

By $\underline{BR}, \overline{BR}$ we define the least and the greatest elements of BR . We say that a (pure strategy) Nash equilibrium of Γ is a profile $f^* \in D$, that it is a fixed point of BR , that is: $f^* \in BR(f^*)$ (see Schmeidler (1973)). To prove existence of such equilibrium we assume:

Assumption 3.1 *For all $(\alpha \in \Lambda, s \in S)$ if $a_n \rightarrow a$ and $f_n \rightarrow f$ pointwise in (α, s) then $r(\alpha, a_n, f_n, s) \rightarrow r(\alpha, a, f, s)$ pointwise in (α, s) . Moreover, r is quasipermodular with $a, (a, s) \rightarrow r(\alpha, a, f, s)$ is $\mathcal{B} \otimes \mathcal{S}$ measurable and has single crossing property in (a, f) .*

Under this assumption we are ready to establish our main theorem of this section.

Theorem 3.1 *Under assumption 3.1 the set of Borel measurable Nash equilibrium is a nonempty with the greatest \overline{f}^* and the least \underline{f}^* elements. Moreover $\overline{f}^* = \lim_n \overline{BR}^n(\overline{d})$ and $\underline{f}^* = \lim_n \underline{BR}^n(\underline{d})$, where $\overline{d}, \underline{d}$ are the greatest and the least elements of D respectively.*

Similarly as above we state a corollary concerning equilibrium comparative statics.

Corollary 3.1 *Assume 3.1 and additionally that r has single crossing properties in (a, s) while $s \rightarrow \tilde{A}(\alpha, s)$ is increasing in Veinott strong set order. Then the greatest and the least Nash equilibria of Γ are isotone on S .*

Few comments concerning theorem 3.1 and corollary 3.1 follow. Firstly, the existence part of this theorem is not new and can be obtained under weaker assumptions (see Balder (1999), Khan (1986) for example). However what is new here, is that we prove existence of the greatest and the least ones. The order on the Schmeidler strategy space is different than for the distributional equilibria. Also the fixed point theorem applied for Schmeidler equilibrium set is different than for Mas-Colell one. It follows from the fact, that the set of (bounded) Borel measurable functions is countably sigma-complete lattice⁵ (for uncountable number of players) only (see Heikkilä and Reffett, 2006, example 2.1.). Secondly to obtain equilibrium existence we need the additional assumption of (pointwise) continuity of the payoff function with strategies of all players. Observe that it is not the case of distributional equilibrium. Thirdly, the (pointwise) computation and equilibrium comparative statics results are similar to those for distributional equilibria in section 2. And fourthly, a careful reader notices the next immediate corollary to our existence result.

⁵I.e., a lattice in which any countable subset has a sup/inf in the set.

Equilibria exists in a class of aggregative games also (see Schmeidler (1973)), i.e. where payoff r depends only on an aggregate φ_f , given by:

$$\varphi_f(s) = \int_{\Lambda} f(\alpha, s) \lambda(d\alpha),$$

or other increasing functionals of f (see also Rath, 1992). We finish with the important remark.

Remark 3.1 *Balder and Rustichini (1994) and Kim and Yannelis (1997) analyze large games without the assumption that the set of players is represented by a measure space and hence without the measurability assumption on a set of players. Our methods can be applied to such (alternative) game/equilibrium definition. The set of Nash equilibria of such a game is then a nonempty complete lattice. In such a case we can also dispense the payoff continuity assumption in 3.1.*

The above remark stresses that assumption that players are represented by a measure space with sigma-algebra (that is countably generated) requires additional continuity type assumption on players payoffs (the feature that is not present in small supermodular games).

We conclude this section with two abstract examples of our methods. The economic applications are moved to the next section. our

Example 3.1 *Let $I = [0, 1]$ with Lebesgue measure, $A = [0, K]$ for $K \in \mathbb{R}_{++}$ and $e_f := \text{ess sup}_{x \in X} f(x)$ and $r(\alpha, a, f, s) = (\alpha + 1)ae_f - a^3$. Here if we compute derivatives we obtain*

$$BR(f)(\alpha) = \min \left(\sqrt{\frac{1}{3}(\alpha + 1)e_f}, K \right).$$

Essential supremum of fixed point must satisfy

$$e_f = \max_{x \in [0, K]} \min \left(\sqrt{\frac{1}{3}(\alpha + 1)e_f}, K \right) = \min \left(\sqrt{\frac{K + 1}{3}} e_f, K \right).$$

Then $f^(\alpha) = 0$ or $f^*(\alpha) = \min \left(\frac{1}{3} \sqrt{(\alpha + 1)(K + 1)}, K \right)$ for $K \geq \frac{1}{2}$ and $f^*(\alpha) = \min \left(\sqrt{\frac{1}{3}(\alpha + 1)K}, K \right)$ if $K \leq \frac{1}{2}$.*

Note that, beside the fact that r does not depend on s , the reward function $f \rightarrow r(\alpha, a, f, s)$ is a composition of some function $F : \mathbb{R}^m \rightarrow \mathbb{R}$ and a functional of f (in this case ess sup). In the next example we do not require this condition.

Example 3.2 *Let $I = [0, 1]$ with Lebesgue measure, let S be an interval in \mathbb{R}^m , and $A(\alpha, s) = [0, s]$. Let $r(\alpha, a, f, s) := \sup_{s' \in S} \{(\alpha + 1)af(\alpha, s') - a^3 - f^3(\alpha, s')\}$. Example satisfies assumption 3.1, but here is not so easy to compute equilibrium algebraically beside some trivial ones.*

4 Symmetric equilibria and uniqueness

Inspired by Mas-Colell (1984), we begin this section with a definition of a game with finite number of player types. Let $\{J_i\}_{i=1}^n$ be a finite collection of disjoint sets of Λ , such that $\bigcup_i J_i = \Lambda$. Let $i = 1, \dots, n$ denote player types, i.e. $\forall i, \forall \alpha, \beta \in J_i, r(\alpha, \cdot) = r(\beta, \cdot)$ and $\tilde{A}(\alpha) \equiv \tilde{A}(\beta)$. In other words, players are identical within the same type i with respect to their payoffs and feasible strategy sets. A distributional game with a finite number of player types Γ_F is defined as in section 2, with the restriction specified above.

Our definition of a distributional symmetric equilibrium follows Mas-Colell (1984). Namely it will be denoted by a measure $\tau \in \mathcal{M}(A)$ defined as in section 2, such that

$$\forall i, \exists a \in A, \tau \left(\left\{ (\alpha, a) \in J_i \times A \mid a \in \arg \max_{a' \in A(\alpha)} r(\alpha, a', \tau) \right\} \right) = \lambda(J_i),$$

where the marginal distribution of τ on Λ is λ . Hence, in a symmetric distributional equilibrium we expect that a.e. player of a given type plays the same strategy. We proceed with the following remark.

Corollary 4.1 *Let assumption 2.1 be satisfied. Then the greatest and the least distributional equilibrium of Γ_F is symmetric.*

Note, that the definition of a symmetric game and equilibrium is different than in Milgrom and Roberts (1990) or Amir, Jakubczyk, and Knauff (2004), who dealt with finite symmetric (quasi)supermodular games. Unlike in the previous papers, we expect players to be symmetric only within a given type i . However, it is straightforward to show that the result holds in particular for games with some i such that $J_i = \Lambda$, and $J_j = \emptyset, j \neq i$, hence satisfying symmetry among all agents. In this case the greatest and the least equilibria are simply Dirac measures concentrated at strategies corresponding to the equilibria.

The symmetry of equilibria can also be extended to large games discussed in section 3. In this case, symmetric equilibrium is a piecewise constant function $f \in \tilde{D}$ defined as in section 3, such that

$$\forall i = 1, \dots, n, \forall \alpha, \beta \in J_i, f(\beta) = f(\alpha) \in \arg \max_{a \in \tilde{A}(\alpha)} r(\alpha, a, f),$$

For convenience we drop parameter s in our notation. However, it is easy to show that all the previous monotone comparative statics results hold in this framework. The following remark is an implication of the previous results.

Remark 4.1 *Let assumption 3.1 be satisfied. Then the greatest and the least Schmeidler-Nash equilibrium of Γ_F is symmetric.*

The proof is analogue to the proof of Corollary 4.1, therefore it is omitted.

The above results have a practical application. Once the existence of symmetric equilibria is determined, the large game can be reduced to a game with finite number of players in the following way. *Step 1:* Construct a game with $n + 1$ players. Endow each of the first n players with a payoff functions $\pi_i : \tilde{A}_i \times \prod_{j=1}^n \tilde{A}_j \rightarrow \mathbb{R}$, where $\tilde{r}_i(a_i, b) := r(\alpha, a_i, \tau(b))$ with $\tilde{A}_i = \tilde{A}(\alpha)$, $\alpha \in J_i$, and $\tau(b)(\cdot) \in \mathcal{D}$ such that⁶ $\tau(b)(\cdot|\alpha) = b_i$ for $i = 1, \dots, n$ and⁷. $\alpha \in J_i$. *Step 2:* Let the $n + 1^{\text{st}}$ player maximize payoff $-\sum_{i=1}^n |a_i - b_i|$ with respect to $\{b_i\}_{i=1}^n$ over $\times_{i=1}^n \tilde{A}_i$.

Note several properties of the game defined above. First of all, construction of τ implies, that as in the corresponding large game, each of the first n players does not observe his impact on the overall distribution of actions, even though there is a finite number of players types. In fact the $n + 1^{\text{st}}$ player is introduced to the game in order to equate his own strategies with those of other players (observe that by construction, it is always feasible and optimal for player $n + 1$ to set $b_i = a_i$), and hence adjust the distribution τ to actions played by the other n agents. This way, any Nash equilibrium of the game defined above, generates distribution⁸ τ , which corresponds to a symmetric distributional equilibrium of the large game. By Lemma 4.1 the greatest and the least element of the equilibrium set is symmetric, therefore under presented assumptions, one can always determine bounds of the equilibrium set of a large game by its finite counterpart.

⁶It means that for $\alpha \in J_i$ the conditional distribution $\tau(b)(\cdot|\alpha)$ is a Dirac delta concentrated at b_i .

⁷Respectively, for Schmeidler-Nash type games payer payoffs would be defined by $\pi(a_i, f) := r(\alpha, a_i, f)$, $\alpha \in J_i$, where $f(\alpha) = \sum_{i=1}^n b_i \chi_{J_i}(\alpha)$ where $\chi_J(\cdot)$ is denoted as an indicator function of the set J .

⁸Respectively, function f .

The above observation is especially useful when approximating equilibria using numerical methods, where the space of agents is discretized. Note however that not every equilibrium can be determined this way, as the game might also have equilibria which are not symmetric.

Another advantage of the finite type approach is that it allows for the usage of tools developed for finite player games analysis. In particular, we are able to determine sufficient conditions for uniqueness of equilibrium.

Our uniqueness result is obtained for large aggregative games. Let $h : \mathcal{M}(A) \rightarrow G$ denote an aggregate function, mapping space of measures τ to a compact set $G \subset \mathbb{R}$. Moreover, assume that payoff of a single player is dependent only on his type, strategy, and the value of the aggregate, i.e. $r : \Lambda \times A \times G \rightarrow \mathbb{R}$. An aggregative distributional equilibrium is a distribution $\tau \in \mathcal{M}(A)$, such that

$$\tau \left(\left\{ (\alpha, a) \in \Lambda \times A \mid a \in \arg \max_{a \in A(\alpha)} r(\alpha, a, h(\tau)) \right\} \right) = 1.$$

Hence, we reproduce definition stated in section 2, with a restriction to the aggregate⁹. In order to make the example complete, we state additional assumptions.

Assumption 4.1 *Assume that*

- $A \subset \mathbb{R}^m$ is endowed with a natural product order and Euclidean topology;
- correspondence \tilde{A} is compact valued, and weakly λ -measurable,
- for λ -a.e. player $r : \Lambda \times A \times G \rightarrow \mathbb{R}$ is continuous and quasisupermodular on A , Borel λ -measurable, and satisfies single crossing property in $(a, g) \in A \times G$,
- for any $g \in G$, $(\alpha, a) \rightarrow r(\alpha, a, g)$ is a Caratheodory function, i.e. is continuous in $a \in A$ and measurable in $\alpha \in \Lambda$,
- $h : \mathcal{M}(A) \rightarrow G$ is increasing.

The above assumption is a slight modification of assumption 4.1, necessary for a specification of a large aggregative game. Moreover, note that the assumption can be easily restated in order to fit the Schmeidler-Nash definition of an equilibrium. We proceed with the following remark.

Remark 4.2 *Let assumption 4.1 be satisfied, then there exists the least and the greatest equilibrium of an aggregative large game.*

The following result provides sufficient conditions for the unique distributional equilibrium.

Theorem 4.1 *Consider an aggregative game with $n < \infty$ player types. Let assumption 4.1 be satisfied. In addition assume that*

- $\tilde{A}_i \subset \mathbb{R}_+^m$ is convex valued,
- for λ -a.e. player r is strictly quasiconcave and twice continuously differentiable on an open set containing A , once continuously differentiable on an open set containing G , with $\forall k = 1, \dots, m, \sum_{j=1}^m \frac{\partial^2 r}{\partial a_k \partial a_j} + \frac{\partial r}{\partial g} < 0$,
- for any symmetric $\tau \in \mathcal{M}(A)$, function $\bar{h} : \times_{i=1}^n \tilde{A}_i \rightarrow G$, $\bar{h}(a) := h(\tau)$, $a \in \times_{i=1}^n \tilde{A}_i$, is well defined and continuously differentiable with $\forall i = 1, \dots, m, \exists M < 1, \frac{\partial \bar{h}}{\partial a_i} \leq M$,

then there exists a unique distributional equilibrium of the game.

⁹Observe, that by defining h as a function mapping \tilde{D} to G , we can define a Schmeidler-Nash counterpart of the aggregative game.

The proof of this result is inspired by the proof of Theorem 2.4 in Curtat (1996) and Gabay and Moulin (1980). It is based on a fact, that a large game with a finite number of types can be represented by its finite counterpart. For this reason it is necessary to impose a relatively strong assumption on the form of an aggregate, which in addition has to be differentiable with respect to the support of the corresponding symmetric distribution. Fortunately, it is possible to determine a broad class of aggregates satisfying our assumption. We present an example of such a function below.

Example 4.1 *Let h map distribution of player/types to an average strategy of players, i.e. $h(\tau) = M \int_A a d\tau(d\alpha \times da)$, $M \in [0, 1)$. Observe, that in case of symmetric distributions, the aggregate takes the form $h(\tau) := M \sum_{i=1}^n \lambda(J_i) a_i$, where a_i is a strategy of type i .*

In this case $\bar{h}(a)$ is differentiable with respect to a , and $\frac{\partial \bar{h}}{\partial a_i} = M \lambda(J_i) < 1$. Hence, assumption of Theorem 4.1 is satisfied.

Having proven the existence and symmetry of distributional equilibria, it is possible to retrieve from it a unique Shmeidler-Nash type equilibrium of a large aggregative game with a finite number of player types. If $\tau^* \in \mathcal{D}$ such that $\tau(\cdot | \alpha) = a_i^*$ for $\alpha \in J_i$ is the unique distributional equilibrium of the game, then

$$f^*(\alpha) := \sum_{i=1}^n a_i^* \chi_{J_i}(\alpha),$$

hence, both equilibria are equivalent, in the sense that one can always be obtained by a transformation of the other.

5 Applications

In this section we discuss three examples of large games where our theorems can be used.

5.1 Product differentiation

Consider the following large game of Bertrand competition with product differentiation (see Allen and Hellwig (1986, 1989) for a related analysis of Bertrand-Edgeworth equilibria in large markets). Let firms be uniformly distributed on $[0, 1]$. Each firm α produces a differentiated product indexed α at price $p_\alpha \in \tilde{A}(\alpha)$. The demand for product is given by a function $Q_\alpha : \tilde{A}(\alpha) \times \tilde{D} \rightarrow \mathbb{R}_+$, \tilde{D} being a set of measurable functions f from Λ to \mathbb{R}_+ with respect to product σ -field, such that $f(\alpha) \in \tilde{A}(\alpha)$, equipped with a pointwise order. As a function of agents, f represents strategies of all players in the market. Each firm operates at a constant marginal cost c_α . Given f , a firm is maximizing $\pi_\alpha(p_\alpha, f) = (p_\alpha - c_\alpha) Q_\alpha(p_\alpha, f)$, with respect to $p_\alpha \in \tilde{A}(\alpha)$. Since any $p_\alpha < c_\alpha$ is strictly dominated by $p_\alpha = c_\alpha$, we restrict $\tilde{A}(\alpha)$ to an interval $[c_\alpha, \bar{p}_\alpha]$, \bar{p}_α being the maximal possible price firm α may quote. For any α , $\tilde{A}(\alpha)$ is a complete lattice.

In order to guarantee, that the best response correspondences are increasing with respect to strategies of other firms, we have to make sure, that for all α , function π_α satisfies the single crossing property (SCP) in (p_α, f) . For this aim we let $\log Q_\alpha$ have increasing differences in (p_α, f) which implies that π_α has SCP. Observe that such property is equivalent to decreasing price elasticity of demand Q_α with respect to f (see Topkis (1995)). In other words, we assume, that the higher the prices of other goods, the more rigid the demand for good α is. Under such assumptions, theorem 3.1 establishes that the set of pure strategy Nash equilibria is nonempty, and possesses the greatest and the least element.

Observe that such result can be obtained for a more general class of cost functions. Let Q_α be decreasing in its first argument, and increasing (respectively: decreasing) in the second. Then, using an argument similar to Milgrom and Shannon (1994) one can show that once a cost function $c_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ of each firm is convex (respectively: concave), π_α satisfies the SCP in (p_α, f) , and so the result of theorem 3.1 also holds.

Eventually, we may determine several results concerning comparative statics of the model. Refer once again to the case of Bertrand competition with constant marginal costs. Since π_α satisfies the single crossing property in (p_α, c_α) , and $[c_\alpha, \bar{p}]$ increases in the strong set order in c_α , due to corollary 3.1, the greatest and the least equilibria of the game are increasing (pointwise) in $\{c_\alpha\}_{\alpha \in [0,1]}$.

In fact our results can be obtained with relatively weak assumptions. The property of log increasing differences of demand is satisfied for a broad class of utility functions. Observe that for CES utility: $\{\int_0^1 x_\alpha^\rho d\alpha\}^{\frac{1}{\rho}}$, where x_α denotes an amount of good α consumed by a consumer with income $I > 0$, and ρ is a parameter, the corresponding demand takes the form of

$$Q_\alpha = \frac{Ip_\alpha^{\frac{1}{\rho-1}}}{\int_0^1 f(\alpha)^{\frac{\rho}{\rho-1}} d\alpha}.$$

Since the measure of a single good is zero, firm α does not observe its influence on the price index $\int_0^1 f(\alpha)^{\frac{\rho}{\rho-1}} d\alpha$, so for any ρ , $\log Q_\alpha$ has constant differences in p_α and prices of the other firms. In such case Bertrand becomes monopolistic competition. Moreover, when demand function is derived through CES utility function, the described competition becomes a large game with an aggregate equivalent to $\int_0^1 f(\alpha)^{\frac{\rho}{\rho-1}} d\alpha$. Finally, the game above is also a special case of a large game, where agents play against a distribution, rather than a function of strategies f . If τ is a distribution over a set of prices, then $\int p^{\frac{\rho}{\rho-1}} \tau(dp) = \int_0^1 f(\alpha)^{\frac{\rho}{\rho-1}} d\alpha$, which gives an alternative specification of the game where results of theorem 2.1 hold.

The above game can be interpreted in several ways, all depending on the definition of payoff functions of players. Once the firm's profit depends solely on the aggregate of actions of other players, as in the example with CES utility function, Bertrand competition becomes a monopolistic competition *à la* Dixit and Stiglitz (1977), where each player does not see his influence on the overall price level in the economy. However, the model can be much more sophisticated. For example, when payoffs are dependent only on strategies of a subset of players in Λ , firms might compete with each other on separate markets, differentiated with respect to location or economic sector. The game might also describe local oligopolies, where firm take into account actions of a measurable subset of players. Thus, the variety of applications is vast.

5.2 Beauty contest

Consider the so called beauty contest analyzed by Acemoglu and Jensen (2010). Suppose that the true beauty of a person is unknown, and each player is a judge with a private signal, given by $\sigma(\alpha) \in \Theta$, where $\Theta \subset \mathbb{R}$ is finite. Let $\sigma : \Lambda \rightarrow \Theta$ denote a function of signals of all players in the game, and let Σ be a space of all functions σ , with a corresponding measure space $(\Sigma, \mathcal{S}, \mu)$, with \mathcal{S} being σ -field and μ an appropriate measure.

Each player makes a public prediction of the true beauty by announcing $x_\alpha(\sigma(\alpha)) \in \Theta$. A set of feasible strategies is therefore a set of measurable functions $x_\alpha : \Theta \rightarrow \Theta$. Every judge is both interested in being close to his private signal, as well as to a distribution of predictions of other players $\tau(\sigma)$, being a measure on Θ , $\forall \sigma \in \Sigma$. We incorporate the space of all τ , denoted by \mathcal{T} , with pointwise (first-order) stochastic ordering. The payoff of each player type α is

$$r_\alpha(x_\alpha, \sigma(\alpha), \tau) = -h_\alpha(|x_\alpha(\sigma(\alpha)) - \sigma(\alpha)|) + \int_\Sigma g_\alpha(|x_\alpha(\sigma(\alpha)) - H_\alpha(\tau(\sigma))|) d\mu(\sigma),$$

where $(\forall \alpha) h_\alpha, g_\alpha : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, $H_\alpha : \mathcal{T} \rightarrow \Theta$ is measurable and increasing, with g convex. Then for any σ ,

$$- [h_\alpha(|x_\alpha(\sigma(\alpha)) - \sigma(\alpha)|) + g_\alpha(|x_\alpha(\sigma(\alpha)) - H_\alpha(\tau(\sigma))|)]$$

satisfies increasing differences in $(x_\alpha(\sigma(\alpha)), \tau(\sigma))$, which implies that r_α satisfies the single crossing property¹⁰ in (x_α, τ) and is quasi-supermodular in x_α , and so assumption 2.1 is satisfied.

By our theorem 2.1, the set of distributional equilibrium is nonempty with a greatest and the least elements. At the same time our approach generalizes the example presented by Acemoglu and Jensen (2010), where H_α is an aggregate, defined by $\int_\Theta \tau(\sigma, s) ds$, while in our case, H_α might be any, increasing function mapping \mathcal{T} to Θ .

Moreover, observe, that H does not have to be defined on a space of distributions. Once H is defined on a pointwise poset of functions f mapping types of players to Θ , the above game can be defined as a generalized large game as in section 3, with the greatest and the least Nash equilibrium.

5.3 Large games with incomplete information and LLN

Consider the following distributional game with incomplete information. Let $(\Lambda, \mathcal{L}, \lambda)$ be a measure space of players. Assume that in the first stage of the game, each player receives a privately known signal $t \in T$, drawn from distribution μ . Let (T, \mathcal{T}, μ) denote the corresponding measure space.

Moreover, given a private signal t , let the payoff be dependent only on distribution of actions of players and their types, i.e. $r : \Lambda \times T \times A \times \tilde{\mathcal{D}} \times \mathcal{M}(T) \rightarrow \mathbb{R}$, where $\mathcal{M}(T)$ is a set of distributions over T , and $\tilde{\mathcal{D}}$ is a set of distributions $\tilde{\tau}_v$ over $\Lambda \times A \times T$, with marginal distribution λ over Λ , and v over T for some $v \in \mathcal{M}(T)$. Therefore, payoff function, for a given measure of types $v \in \mathcal{M}(T)$ takes values $r(\alpha, t, a, \tilde{\tau}_v, v)$. In addition, denote set of feasible actions of player α , with a private signal t , by $\tilde{A}(\alpha, t)$.

Define a family of random variables $\{t_\alpha | \alpha \in \Lambda\}$, where t_α is drawn from distribution μ . Let (V, \mathcal{V}, Ψ) be a measure space over all possible distributions of $\{t_\alpha | \alpha \in \Lambda\}$. Then, interim payoff of agent α , given his type t would be

$$\int_V r(\alpha, t, a, \tilde{\tau}_v, v) d\Psi(v).$$

The above specification poses several substantial issues. First of all, one is obliged to define measure space (V, \mathcal{V}, Ψ) , which is a non-trivial task, concerning that it has to be defined over a space of measures. This concerns in particular the existence of a non-trivial measure Ψ over a sensible σ -algebra \mathcal{V} , with a appealing economic interpretation. Moreover, in order for the problem to be well defined, one should also impose a measurable structure on $\tilde{\mathcal{D}}$, so that the above integral was well defined. Both issues are equally challenging.

The economic intuition suggests however, that the above problem could be simplified by assuming that a form of the strong law of large numbers holds. In this case, the distribution of types would be equivalent to μ , i.e. $\mu(\{t_\alpha \in D\}) = \lambda(\{\alpha \in \Lambda | t_\alpha \in D\})$, and the problem would become deterministic.

Although this approach is appealing, it has several drawbacks. First of all, it is a well known fact that in general a continuum of random variables drawn independently from an identical distribution does not satisfy the strong law of large numbers (see Feldman and Gilles (1985), Judd (1985); see also Alós-Ferrer (1998) for discussion). Even though there are several examples of distributions preserving the property for a continuum of random variables, they are either constructed arbitrarily with a substantial degree of freedom (see Judd (1985)), or require a certain level of abstraction of the space of players, making it hard to interpret¹¹ (see Green (1994), Sun (2006)).

Fortunately, once the independence assumption is abandoned, one can construct a distribution which satisfies the strong law of large numbers. Feldman and Gilles (1985), in Proposition 2

¹⁰Observe that a sum of function with increasing differences, has also increasing differences, see Lemma 7.3 in the Technical Appendix.

¹¹The space of players might be either endowed with a σ -algebra which is not countably generated, hence is not Borel, like in Green (1994), or cardinality of the player set might be larger than continuum, as in Sun (2006).

of their work, present a distribution which assigns *a priori* identically distributed, but correlated outcomes to the population, while Alós-Ferrer (1998) extend their idea to a more general class of distributions.

Obviously lack of independence might be a restrictions in some some situations, but as Alós-Ferrer (1998) observes "frequently, independence is added to the hypothesis of the model because it is thought to be necessary for the application of an unspecified law of large numbers". Moreover, as Al-Najjar (1995) points out, "strong distributional assumptions should be avoided in the study of large games where correlation due to sunspots, correlated types, or correlated randomization devices is quite natural". This can be related to our example of large anonymous Bayesian games, where correlated signals might not only be a natural assumption, but also allow to incorporate tools developed in section 2 to the study of large economies with incomplete information.

Returning to our original problem, assume that $\mu(\{t_\alpha \in D\}) = \lambda(\{\alpha \in \Lambda | t_\alpha \in D\})$, for any $D \in \mathcal{T}$, hence t_α satisfies a form of the strong law of large numbers. By the work of Feldman and Gilles (1985) as well as Alós-Ferrer (1998), we know that such random variables exist, although they might not be independent. Moreover, let assumption 2.1 be satisfied with r being measurable in t , and $\tilde{A}(\alpha, \cdot)$ compact, complete lattice valued and weakly measurable in (α, t) .

Observe, that once the above condition is satisfied, the optimization problem of each individual becomes deterministic. Since the distribution of types is known, each agent has to solve his own maximization problem of the following form,

$$\max_{a \in \tilde{A}(\alpha, t)} r(\alpha, t, a, \tilde{\tau}_\mu, \mu),$$

given $\tilde{\tau}_\mu$. In fact we can easily redefine the game presented in section 2, so that all the results would hold for games with incomplete information. Let $\varepsilon = (\alpha, t) \in \Lambda \times T$ denote the "name" of a player. By the above assumptions we conclude that the distribution of ε is a product measure $\lambda \otimes \mu$. Then, the optimization problem presented above can be restated in the following form

$$\max_{a \in \tilde{A}(\varepsilon)} r_\mu(\varepsilon, a, \tilde{\tau}_\mu),$$

where r_μ varies with respect to a given distribution of types, but at the same time takes the form of payoff discussed in section 2. Analogously, an equilibrium would be a distribution $\tilde{\tau}_\mu$, defined as in section 2, with a marginal distribution on $\Lambda \times T$ equivalent to $\lambda \otimes \mu$. Hence, all the results presented in the section, would hold.

The fact that μ satisfies the law of large numbers, enable us to extend the space of player "names", so that it would incorporate not only α , but also the private signal t . This way, the whole game becomes deterministic not only on the aggregated level, as it usually takes place in large economies with uncertainty, but also on the individual level. Since each agent takes into the account only the distribution of signals, once μ is known, uncertainty vanishes. At the same time the assumptions that a type of each agent is a random variable correlated with types of other players seems to be not that restrictive. In fact, we claim that it might be desired from an economic point of view, where sunspots, correlated randomizations take place.

5.4 General equilibrium with gross substitutes

We explore an example presented for a discrete goods economy in Milgrom and Shannon (1994). Consider an economy with a continuum of goods characterized by a measure space $(\Lambda, \mathcal{L}, \lambda)$, where $\alpha \in \Lambda$ shall denotes a type of good. Let $p : \Lambda \rightarrow \mathbb{R}_+$ be a measurable price function, where $p(\alpha)$ describes a price of good α . Moreover, we assume, that there exists an α^* , such that $p(\alpha^*) = 1$. We shall refer to good α^* as to a numeraire. We incorporate the space of all such functions with a pointwise order. Define $d_\alpha : \mathbb{R}_+ \times \tilde{D} \times \Theta \rightarrow \mathbb{R}$, taking values $d_\alpha(p(\alpha), p, \theta)$ as

an excess demand function for good α , where \tilde{D} is a set of measurable functions p , and θ is a parameter belonging to a poset Θ . Assume that for any p there exist a such that d_α is zero¹².

Observe, that in order to determine general equilibrium of the economy, one can analyse a game among an infinite number of market makers, each responsible for a market of a given type α . The aim of each player is to maximize $r_\alpha(a, p, \theta) = -|d_\alpha(a, p, \theta)|$ with respect to price a . Once $r_\alpha(a, p, \theta)$ exhibits the SCP in a, p , the equilibrium prices are determined by a Nash equilibrium of the large game. One way to guarantee that $r_\alpha(a, p, \theta)$ satisfies the SCP is to limit the analysis to economies with gross substitutes. By the work of Arrow and Hurwicz (1958) and Arrow, Block, and Hurwicz (1959) we conclude that if economy is characterized by gross substitutes, then for any α , $d_\alpha(a, p, \theta)$ is strictly decreasing in the first argument, and strictly increasing in the second. Under the above assumptions, r_α satisfies the (strict) single crossing property in (a, p) . Therefore, by theorem 3.1, there exists a non-empty set of Nash equilibria, with the greatest and the least element. At the same time, each element of the set is a measurable price function characterizing general equilibrium in the underlying economy.

Similarly to the discrete case, in a continuous goods space, the normalized equilibrium is also unique. Let \bar{p} , and \underline{p} denote the greatest and the least equilibrium prices. If the equilibrium is not unique, then $\bar{p} \geq \underline{p}$. Observe, that in case of the numeraire good $d_{\alpha^*}(1, \bar{p}, \theta) = d_{\alpha^*}(1, \underline{p}, \theta) = 0$, which yields a contradiction, since d is strictly increasing in its second argument.

In addition, we can determine several results concerning comparative statics of the equilibrium. Observe, that once d is increasing with parameter θ , r_α satisfies the single crossing property in (a, θ) which implies, that the equilibrium price function is increasing (pointwise) in θ . Finally note, that the demand function derived from a CES utility function (as in example 5.1) generates an excess demand function which satisfies our assumptions. This implies, that in some cases, one might construct a game, where market makers would maximize their payoffs taking a distribution of prices τ , or some aggregate value, rather than price function p as given.

6 Conclusion

In this paper we prove existence of distributional and Nash equilibria in the class of large games with complementarities under rather general conditions. Moreover we provide tools for equilibrium computation and equilibrium comparative statics. The techniques of the paper can be generalized, however, in the few directions that we discuss here.

Firstly using the monotone comparative statics of Veinott (1992) or Calzi and Veinott (1992) it should be possible to weaken the single crossing assumptions to the more general ones, but still allowing for an existence of measurable and increasing selections. Secondly, using results of Quah (2007) it should be possible to analyze large games with aggregative constraints. Thirdly using Heikkilä and Reffett (2006) fixed point results for set-valued mappings in products of posets, should suffice to generalize the action spaces used by all players. Finally using recent result for dynamic supermodular stochastic games (see Sleet (2001) or Balbus, Reffett, and Woźny (2010)) we should be able to complement Jovanovic and Rosenthal (1988) or Chakrabarti (2003) existence theorems for large anonymous dynamic games with computational results. The last extension is nontrivial, as it involves the law of large numbers for a continuum of random variables; a theorem closely related to the existence and purification of equilibrium in large games (see Pascoa, 1998).

On the other hand there is a set of open question that shall be answered. Specifically: can our equilibrium existence and characterization results be generalized (i) in the line of Khan, Rath, and Sun (1997) to allow for infinite-dimensional spaces of actions, and (ii) to allow for Bayesian games (see Balder and Rustichini, 1994, Kim and Yannelis, 1997). Given the order theoretical tools applied in our paper and results for small Bayesian supermodular games (see Van Zandt

¹²Conditions: d_α continuous, homogeneous of degree zero for every α , and bounded from below in the two first arguments; moreover $\{d_\alpha(a, p, \theta)\}_{\alpha \in \Lambda}$ satisfy Walras law, and if $p^n \rightarrow p$, where $p \neq 0$ and $p(\alpha) = 0$ for some α , then $\max_{\alpha \in \Lambda} \{d_\alpha(p(\alpha), p)\} = \infty$ should suffice.

(2010), Vives and Van Zandt (2007)), we think that at least some of these generalizations should be possible.

7 Technical appendix

7.1 Auxiliary results

Markowsky (1976) proved the following:

Theorem 7.1 (*Markowsky (1976)*) *Let $f : X \rightarrow X$ be isotone and X a chain complete poset then fixed points set of f is a chain complete poset. Moreover $\bigvee \{x : x \leq f(x)\}$ is the greatest fixed point and $\bigwedge \{x : x \geq f(x)\}$ is the least fixed point of f .*

We now state the Tarski-Kantorovitch fixed point theorem and our generalization of this result in terms of fixed-point comparative statics.

Definition 7.1 *A function $F : X \rightarrow X$ is monotonically-sup-preserving, if for any monotone sequence $\{x_n\}_{n=0}^{\infty}$ we have: $F(\bigvee x_n) = \bigvee F(x_n)$. We define monotonically inf-preserving functions analogously. F is said to be monotonically-sup-inf-preserving if and only if, it is both monotonically-sup and monotonically-inf-preserving.*

The Tarski-Kantorovitch (see Dugundji and Granas, 1982, thm 4.2) theorem assures that:

Theorem 7.2 *Let X be a countably chain complete poset with the greatest \bar{x} and the least element \underline{x} respectively. Let $F : X \rightarrow X$ be an increasing function. Then*

- (i) *if F is monotonically-inf-preserving then $\bigwedge F^n(\bar{x})$ is the greatest fixed point of F ,*
- (ii) *if F is monotonically-sup-preserving then $\bigvee F^n(\underline{x})$ is the least fixed point of F .*

In Balbus, Reffett, and Woźny (2011) we proved the following auxiliary result.

Theorem 7.3 *Let $F : X \times T \rightarrow X$ be an increasing operator (with respect to product order on $X \times T$), where T is a poset and X is a countably chain complete poset with the greatest and least elements. If for any $t \in T$ function $F(\cdot, t)$ is monotonically-sup-inf preserving then the fixed point set of $F(\cdot, t)$ denoted by $\Phi(t)$ is a countably chain complete poset. Moreover the least and the greatest fixed point selections $t \rightarrow \underline{\Phi}(t)$ and $t \rightarrow \bar{\Phi}(t)$ are increasing.*

We now proceed to the proof of this theorem in two lemmas. Let (X, \leq) be a countably chain complete poset. It implies that each increasing sequence has a supremum, and each decreasing sequence has infimum. Assume that X has the greatest element \bar{x} and the least element \underline{x} . For a monotone sequence $\{x_n\}_{n=0}^{\infty}$, let

$$\bigvee x_n := \sup_{n \in \mathbb{N}} x_n,$$

and

$$\bigwedge x_n := \inf_{n \in \mathbb{N}} x_n.$$

Denote by $F^n(x)$ the n -th orbit (or iteration) of x under the function F , i.e. $F^n(x) = F \circ F \circ \dots \circ F(x)$. We have the following two lemmas, with the first pertaining to fixed-point set characterization, and the second pertaining to fixed-point comparative statics.

Lemma 7.1 *Let X be a countably chain complete poset and $F : X \rightarrow X$ an increasing function, that is monotonically sup-inf-preserving. Then the set of fixed points is a nonempty countably chain complete poset with*

$$\bar{\Phi} = \bigvee \{x : F(x) \geq x\}, \tag{1}$$

and

$$\underline{\Phi} = \bigwedge \{x : F(x) \leq x\}. \tag{2}$$

Proof of lemma 7.1: By Tarski-Kantorovitch theorem fixed F has fixed points. Take e_n , an increasing set of fixed points. Let $\bar{e} = \bigvee e_n$. Then,

$$\begin{aligned} F(\bar{e}) &= F\left(\bigvee e_n\right), \\ &= \bigvee F(e_n), \\ &= \bigvee e_n = \bar{e}. \end{aligned}$$

Similarly, we prove the thesis for decreasing sequences. Now, we finally prove equality (1). Let x be arbitrary point such that $x \leq F(x)$. Clearly $x \leq \bar{x}$. Assume $x \leq F^n(\bar{x})$. Then, $x \leq F(x) \leq F(F^n(\bar{x})) = F^{n+1}(\bar{x})$. Hence, $x \leq \bar{\Phi}$. Since $\bar{\Phi} \in \{x : F(x) \geq x\}$, equality (1) is proven. We prove (2) analogously. ■

We finally prove a theorem (and a corollary) on increasing selections for parameterized problems that we use in the paper to obtain our results on equilibrium monotone comparatives.

Lemma 7.2 *Let X be a countably chain complete poset with the greatest and least elements and T a poset. If $F : X \times T \rightarrow X$ is increasing, and monotonically-sup-inf preserving on X then $t \rightarrow \bar{\Phi}(t)$ and $t \rightarrow \underline{\Phi}(t)$ are isotone.*

Proof of lemma 7.2: Let $t_1 \leq t_2$. From theorem 7.1 we know that $m_i := \bar{\Phi}(t_i) = \bigvee \Gamma_i := \bigvee \{x : F(x, t_i) \leq x\}$. Note that by isotonicity of $F(x, \cdot)$ we obtain $m_1 = F(m_1, t_1) \leq F(m_1, t_2)$. Hence $m_1 \in \Gamma_2$. Since m_2 is the greatest element of Γ_2 , hence $m_1 \leq m_2$. ■

7.2 Proofs of main results

Proof of lemma 2.1: *Step 1.* We show that operator \bar{T} is well defined (the same argument can be used for \underline{T}). Since $a \rightarrow r(\alpha, a, \tau)$ is quasimodular and has a single crossing properties in (a, τ) hence by Milgrom and Shannon (1994) or Veinott (1992) generalization of Topkis (1978) monotonicity theorem, its set of maximizers is a complete sublattice for fixed τ . Moreover, the set of maximizers is increasing in the Veinott strong set order, whenever τ increase. Hence the set of maximizers is nonempty and increase with respect to τ .

We show that it has increasing and measurable selector. Since $r(\alpha, a, \tau)$ is continuous in a and measurable in τ , and λ is a finite measure, hence r is a Carathéodory function for all τ . At the same time $\tilde{A}(\alpha)$ is weakly measurable, hence by Theorem 18.19 in Aliprantis and Border, $R_\tau(\alpha) := \max_{a \in \tilde{A}(\alpha)} r(\alpha, a, \tau)$ is measurable function and argmax correspondence $m(\alpha, \tau)$

is measurable in α hence also weakly measurable. Observe that $\bar{m}(\alpha, \tau) = (\hat{a}_1, \dots, \hat{a}_m)$ where $\hat{a}_i = \max_{a \in m(\alpha, \tau)} Proj_i(a)$ and $Proj_i$ means projection of a vector on its i -th coordinate¹³ Since projection is a continuous function, hence again by Maximum Measurable Theorem $\bar{m}(\alpha, \tau)$ is λ -measurable function. Hence \bar{T} is well defined and maps \mathcal{D} into itself.

Step 2. As a result, for arbitrary $\alpha \in \Lambda$ we have $\bar{m}(\alpha, \tau_2) \geq \bar{m}(\alpha, \tau_1)$ whenever $\tau_2 \succeq_P \tau_1$. We

¹³For $a_i \in \mathbb{R}^m$ by max we denote a max of all coordinates.

show that $\bar{T}(\tau_2) \succeq \bar{T}(\tau_1)$. Let $f : \lambda \times A \rightarrow \mathbb{R}$ be a increasing and measurable function. Then

$$\begin{aligned}
\int_{\Lambda_0 \times A_0} f(\alpha, a) \bar{T}(\tau_2)(d\alpha \times da) &= \int_{\Lambda_0} f(\alpha, \bar{m}(\alpha, \tau_2)) \lambda(d\alpha) \\
&\geq \int_{\Lambda_0} f(\alpha, \bar{m}(\alpha, \tau_1)) \lambda(d\alpha) \\
&= \int_{\Lambda_0 \times A_0} f(\alpha, a) \bar{T}(\tau_1)(d\alpha \times da)
\end{aligned} \tag{3}$$

where first equality follows by definition of \bar{T} , (3) follows since both f and $\bar{m}(\alpha, \cdot)$ are increasing. We use the similar argument to prove monotonicity of \underline{T} . \blacksquare

Proof of theorem 2.1: *Step 1.* We show the existence of the greatest equilibria. Analogously we are able to prove existence of the least equilibrium. Observe that $\tilde{A}(\alpha)$ is compact valued, hence from Lemma 2.2 \mathcal{D} is a chain complete poset. Since from Lemma 2.1 \bar{T} is a monotone operator mapping \mathcal{D} into itself, hence by Theorem 7.1 we conclude that set of fixed points of \bar{T} has the greatest element. Let τ^* be such a point. Then

$$\tau^*(\{(\alpha, a) : a \in m(\alpha, \tau^*)\}) \geq \tau^*(\{(\alpha, a) : a = \bar{m}(\alpha, \tau^*)\}) = 1.$$

Hence τ^* is a distributive equilibrium.

Step 2. We show that τ^* is the greatest distributional equilibrium. Let τ_0 be some other equilibrium. Then by definition of distributional equilibria

$$\begin{aligned}
1 &= \tau_0(\{(\alpha, a) : a \in m(\alpha, \tau_0)\}) \\
&\leq \tau_0(\{(\alpha, a) : a \leq \bar{m}(\alpha, \tau_0)\}).
\end{aligned}$$

Hence τ_0 is concentrated in the set $E_0 = \{(\alpha, a) : a \leq \bar{m}(\alpha, \tau_0)\}$.

Hence taking an increasing function $f : \Lambda \times A \rightarrow \mathbf{R}$ we have

$$\begin{aligned}
\int_{\Lambda \times A} f(\alpha, a) \tau_0(d\alpha \times da) &= \int_{E_0} f(\alpha, a) \tau_0(d\alpha \times da) \\
&\leq \int_{\Lambda} f(\alpha, \bar{m}(\alpha, \tau_0)) \tau_0(d\alpha \times da) \\
&= \int_{\Lambda} f(\alpha, \bar{m}(\alpha, \tau_0)) \lambda(d\alpha) \\
&= \int_{\Lambda} f(\alpha, a) \bar{T}(\tau_0)(d\alpha \times da)
\end{aligned} \tag{4}$$

where (4) follows by definition of E_0 and the last equation from definition of \bar{T} . Therefore $\tau_0 \preceq_P \bar{T}(\tau_0)$. By Lemma 2.1 \bar{T} is increasing, hence by theorem 7.1 $\tau^* \succeq_P \tau_0$ and hence τ^* is the greatest distributional equilibrium.

Step 3. We show that under additional continuity assumption \bar{T} is monotonically inf-preserving. Let $\tau^n \in \mathcal{D}$ be a sequence, monotonically decreasing with infimum τ (this limit exists, as by Lemma 2.2 set of distributions in \mathcal{D} is a chain complete). Then

$$r(\alpha, \bar{m}(\alpha, \tau^n), \tau^n) \geq r(\alpha, a, \tau^n).$$

Since $\bar{m}(\alpha, \tau^n) \in \tilde{A}(\alpha)$ which is a compact subset, hence the limit of this sequence say m_0 (exists because by lemma 2.2 distributions forms continuous chain complete poset) satisfies $m_0 \in \tilde{A}(\alpha)$. By continuity of r we have then $r(\alpha, m_0, \tau) \geq r(\alpha, a, \tau)$ for all $a \in \tilde{A}(\alpha)$. Therefore $m_0 \in m(\alpha, \tau)$. Hence

$$m_0 \leq \bar{m}(\alpha, \tau). \quad (5)$$

On the other hand, since $\tau \preceq_P \tau^n$ for all n , hence by monotonicity of \bar{m} $\bar{m}(\alpha, \tau) \leq \bar{m}(\alpha, \tau^n)$ and then

$$\bar{m}(\alpha, \tau) \leq \lim_{n \rightarrow \infty} \bar{m}(\alpha, \tau^n) = m_0. \quad (6)$$

Combining (5) and (6) we have desired equality that is $\bar{m}(\alpha, \tau) = \lim_{n \rightarrow \infty} \bar{m}(\alpha, \tau^n)$. Hence $\bar{T}(\cdot)$ is monotonically inf-preserving. By definition \bar{T} is also monotonically inf-preserving.

Step 4. We show that $\tau^* = \lim_{n \rightarrow \infty} \bar{T}^n(\delta)$. By Step 1 \bar{T} is monotonically inf preserving. Therefore, (by Lemma 2.1 \bar{T} is increasing) the convergence of $\bar{T}^n(\delta) \rightarrow \tau^*$ holds by theorem 7.2. Similarly we construct the least distributional equilibrium using \underline{T} .

Step 5. If $\bar{T} = \underline{T} := T$ then T is monotonically sup-inf preserving and hence by Theorem 7.1 the fixed point set of T is a chain complete poset. ■

Proof of theorem 2.2: *Step 1.* First we show that the greatest and the least equilibria are isotone in s . We show this property for the greatest equilibrium. Analogously we show it for the least equilibrium.

As previously define $m_s(\alpha, \tau)$ for $s \in S$ and $\bar{m}_s(\alpha, \tau)$. Let us also define \bar{T}_α^s as a Dirac delta concentrated in $\bar{m}_s(\alpha, \tau)$ and \bar{T}^s . We need to show that $s \rightarrow \bar{T}^s$ is isotone. Note that the set of maximizers of $a \rightarrow r(\alpha, a, \tau, s)$ is increasing in the Veinott order set with respect to s . Therefore, $\bar{m}_s(\alpha, \tau)$ increase in s and hence \bar{T}^s increase in s as well. Counterparting Theorem 7.3 for chain complete posets, this implies that maximal equilibrium in $\Gamma(s)$ $\bar{\tau}_s^*$ is \succeq_P increasing with respect to s .

Step 2. Similarly as in the proof of Theorem 2.1 we show that $(\alpha, s) \rightarrow \bar{m}_s(\alpha, \tau)$ is $\mathcal{L} \otimes \mathcal{S}$ measurable whenever $\tau^s \in \mathcal{D}_s$. This implies $\bar{\tau}_s^*$ is $\mathcal{L} \otimes \mathcal{S}$ measurable in (α, s) . ■

Proof of theorem 3.1: Note that since $A(\alpha, s)$ is compact hence $\arg \max_{a \in A(\alpha, s)} r(\alpha, a, f(\cdot, s), s)$

is nonempty. Moreover, r is quasisupermodular in a and has single crossing properties in (a, f) . Since $A(\alpha, s)$ is a lattice and $\mathcal{M}(\mathcal{L} \otimes \mathcal{S})$ is a poset, by Milgrom and Shannon (1994) or Veinott (1992) generalization of Topkis (1978) monotonicity theorem, the set of maximizers is a complete sublattice for fixed f . Since $BR(f)(\alpha, s)$ is a complete sublattice valued, hence there exist functions $\bar{f} := \bigvee BR(f)$ and $\underline{f} := \bigwedge BR(f)$. Note that by above mentioned theorems both $\bar{f}, \underline{f} \in BR(f)$ are increasing as a functions of f .

Note that generally correspondence $BR(f)(\alpha, s)$ may contain unmeasurable functions. We now show that $(\alpha, s) \rightarrow \bar{f}(\alpha, s)$ and $(\alpha, s) \rightarrow \underline{f}(\alpha, s)$ are measurable. By Measurable Maximum Theorem (see e.g. theorem 17.18 in Aliprantis and Border (1999)) $\max_{a \in A(\alpha, s)} r(\alpha, a, \varphi_f(s), s)$ is

measurable on Λ , and argmax correspondence $(\alpha, s) \rightarrow BR(f)(\alpha, s)$ is measurable for all $f \in D$. Finally consider $\bar{f} := (\bar{f}_1, \dots, \bar{f}_m)$. To see that each $(\alpha, s) \rightarrow \bar{f}_j(\alpha, s)$ is measurable observe that $\bar{f}_j(\alpha, s) = \max_{a \in BR(f)(\alpha, s)} a_j$ and hence by Measurable Maximum Theorem function $\alpha \rightarrow \bar{f}_j(\alpha, s)$ is measurable for any $j = 1, \dots, m$.

As \bar{f} and \underline{f} are measurable, $\overline{BR} := \bigvee BR$ and $\underline{BR} = \bigwedge BR$ map D into D , which is a σ -complete lattice. Moreover by our continuity assumptions \overline{BR} is monotonically inf preserving. Hence and by our theorem 7.3 both operators have the greatest and least fixed points respectively.

Denote the greatest fixed point of \overline{BR} by \overline{f}^* and the least of \underline{BR} by \underline{f}^* . Then if f_0 is an arbitrary Nash equilibrium, then and by theorem 7.3 we have $\underline{f}^* = \bigwedge \{f : \underline{BR}(f) \leq f\} \leq f_0 \leq \bigvee \{f : \overline{BR}(f) \leq f\} = \overline{f}^*$, ■

Proof of corollary 4.1: Let $\overline{\tau}^*$ denote the greatest distributional equilibrium of Γ_F . By theorem 2.1 it exists. Observe, that

$$\forall i = 1, 2, \dots, n, \forall \alpha, \beta \in J_i, \overline{m}(\alpha, \overline{\tau}^*) = \overline{m}(\beta, \overline{\tau}^*),$$

Therefore $\overline{T}(\overline{\tau}^*)$, where \overline{T} is defined as in section 2, is symmetric. By definition $\overline{T}(\overline{\tau}^*) = \overline{\tau}^*$, which implies that $\overline{\tau}^*$ must be symmetric. We use an analogue argument to prove the result for the least equilibrium. ■

Proof of theorem 4.1: By Remark 4.2 the greatest and the least equilibrium of the game exist. Moreover, by Remark 4.1, they are both symmetric. Hence, it is sufficient to show that the game has a unique symmetric distributional equilibrium. We divide our proof into following steps.

Step 1. Define a finite game with $n + 1$ players in the following way. $\forall i = 1, \dots, n, \pi_i(a_i, b) = r(\alpha, a_i, \overline{h}(b)), \alpha \in J_i, a_i \in A, b \in A^n$. Each of the first n players maximizes r_i with respect to a_i over $\tilde{A}_i = \tilde{A}(\alpha), \alpha \in J_i$. The $n + 1^{\text{st}}$ player payoff is then $-\sum_{i=1}^n |a_i - b_i|, b_i \in \tilde{A}_i$, which is maximized with respect to b over $\times_{i=1}^n \tilde{A}_i$. Endow A with a *taxicab* norm $\|\cdot\|_1$, and $\times_{i=1}^n \tilde{A}_i$ with a n times product of this norm.

Step 2. Consider one of the first n players, say player i . Let $x_i(b) := \arg \max_{a \in \tilde{A}_i} \pi_i(a, b)$. Since r is quasiconcave in a , continuous in b , and \tilde{A}_i is compact and convex, x_i is a continuous function (see Berge (1997) Maximum Theorem). Moreover, due to assumption 4.1 concerning quasisupermodularity of r , x_i is also increasing (see e.g. Milgrom and Shannon (1994)).

Step 3. Define $\tilde{\pi}_i(y, b) := \pi_i(M\phi(b)\mathbf{1} - y, b)$, where $\phi(b) = \sum_{i=1}^n \sum_{j=1}^m b_{ij}, b_i \in A, b_{ij} \in \mathbb{R}$ is the j^{th} coordinate of $b_i, y \in A$, and $\mathbf{1}$ is a unit vector in \mathbb{R}^m . Now we will show that $\tilde{\pi}_i(y, b)$ has increasing differences in (y, b) .

Let y_j be the j^{th} coordinate of y . Observe, that $\frac{\partial \tilde{\pi}_i}{\partial y_j} = -\frac{\partial r}{\partial a_j}(\alpha, \cdot)$. Therefore

$$\begin{aligned} \frac{\partial^2 \tilde{\pi}_i}{\partial y_j \partial b_k} &= -M \frac{\partial \phi}{\partial b_k} \sum_{l=1}^m \frac{\partial^2 r}{\partial a_j \partial a_l}(\alpha, \cdot) - \frac{\partial r}{\partial g}(\alpha, \cdot) \frac{\partial \overline{h}}{\partial b_k} \\ &\geq -M \left(\sum_{l=1}^m \frac{\partial^2 r}{\partial a_j \partial a_l}(\alpha, \cdot) + \frac{\partial r}{\partial g}(\alpha, \cdot) \right) \geq 0, \end{aligned} \quad (7)$$

$\forall j, k$, where $\alpha \in J_i$. Since \tilde{A}_i is a compact, convex subset of \mathbb{R}_+^m , denote it by $\tilde{A} \equiv [0, \tilde{a}_i]$, where $\tilde{a}_i \in \mathbb{R}_+^m$. The set of feasible y , given b is therefore defined by $[M\phi(b)\mathbf{1} - \tilde{a}_i, M\phi(b)\mathbf{1}]$, which is ascending in the Veinott strong set order. Therefore, by Theorem 6.2 in Topkis (1978),

$$y_i^*(b) = \arg \max_{y \in [M\phi(b)\mathbf{1} - \tilde{a}_i, M\phi(b)\mathbf{1}]} \tilde{\pi}_i(y, b)$$

is ascending in the Veinott strong set order. Since $x_i(b)$ is unequivocally defined by $y_i^*(b)$, y_i^* is a function. Denote the j^{th} coordinate of $x_i(b)$ and $y_i^*(b)$ respectively by $x_{ij}(b)$ and $y_{ij}^*(b)$. By definition of $y_i^*(b)$, for any $b' \geq b, \forall j = 1, \dots, m$,

$$0 \leq x_{ij}(b') - x_{ij}(b) \leq M(\phi(b') - \phi(b)) = M\|b' - b\|_1,$$

and $x_{ij}(b) - x_{ij}(b') \geq -M\|b' - b\|_1$. For any two unordered b' and b ,

$$0 \leq x_{ij}(b') - x_{ij}(b) \leq x_{ij}(b' \vee b) - x_{ij}(b' \wedge b) \leq M\|b' - b\|_1,$$

since $\|b' - b\|_1 = \|b' \vee b - b' \wedge b\|_1$. Similarly, $x_{ij}(b) - x_{ij}(b') \geq -M\|b' - b\|_1$. Therefore, since $M < 1$, $\forall i, x_i$ is Lipschitz with modulus $M < 1$.

Step 4. Define operator $T : \times_{i=1}^n \tilde{A}_i \rightarrow \times_{i=1}^n \tilde{A}_i$ by $T(b) := \times_{i=1}^n x_i(b)$. Since $\forall i, x_i$ is Lipschitz with modulus M , T is a contraction. Moreover, $\times_{i=1}^n \tilde{A}_i$ is a closed subset of a Banach space \mathbb{R}^{mn} . By Banach Fixed Point Theorem, T has a unique fixed point, denote it a^* . By definition $(a = a^*, b = a^*)$ is the unique Nash equilibrium of the game defined in Step 1.

Step 5. Note that τ^* satisfying $\tau^*(\cdot|\alpha) = a_i^*$ for $\alpha \in J_i$ for $i = 1, \dots, n$ is the unique symmetric distributional equilibrium of the large aggregative game with n types. Since the greatest and the least equilibria are symmetric, they must be equivalent. The proof is complete. ■

Lemma 7.3 *Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be a decreasing, concave function. Then, function $g : X \times S \rightarrow \mathbb{R}$, such that $X, S \in \mathbb{R}$, X is convex, and $g(x, s) := f(|x - s|)$, satisfies SCP in (x, s) .*

Proof of lemma 7.3: By concavity and monotonicity of f , as well as convexity of $|\cdot|$, we conclude that for any convex set $X \in \mathbb{R}$, function $h := f \circ |\cdot| : X \rightarrow \mathbb{R}$ is concave. Hence, for any $x' \geq x$, $x', x \in X$, and $s' \geq s$, such that h is well defined,

$$\frac{h(x' - s') - h(x - s')}{(x' - s') - (x - s')} \geq \frac{h(x' - s) - h(x - s)}{(x' - s) - (x - s)},$$

which implies that $h(x' - s') - h(x - s') \geq h(x' - s) - h(x - s)$. Since $g(x, s) = h(x - s)$, the proof is complete. ■

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