A Constructive Study of Markov Equilibria in Stochastic Games with Strategic Complementarities*

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Abstract

We study a class of discounted infinite horizon stochastic games with strategic complementarities. We first characterize the set of all Markovian equilibrium values by developing a new Abreu, Pearce, and Stacchetti (1990) type procedure (monotone method in function spaces under set inclusion orders). Secondly, using monotone operators on the space of values and strategies (under pointwise orders), we prove existence of a Stationary Markov Nash equilibrium via constructive methods. In addition, we provide monotone comparative statics results for ordered perturbations of the space of stochastic games. Under slightly stronger assumptions, we prove the stationary Markov Nash equilibrium values form a complete lattice, with least and greatest equilibrium value functions being the uniform limit of successive approximations from pointwise lower and upper bounds. We finally discuss the relationship between both monotone methods presented in a paper.

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1 Introduction and related literature

Since the class of discounted infinite horizon stochastic games was introduced by Shapley (1953), and subsequently extended to more general n-player settings (e.g., Fink (1964)), the question of existence and characterization of (stationary) Markov Nash equilibrium (henceforth, (S)MNE) has been the object of extensive study in game theory.[1] Further, and perhaps more central to motivation of this paper, in recent times stochastic games have become a fundamental tool for studying dynamic economic models, where agents possess some form of limited commitment. Examples of such limited commitment frictions are extensive, as they arise naturally in diverse fields of economics, including work in: (i) dynamic political economy (e.g., see Lagunoff (2009), and reference contained within), (ii) dynamic search with learning (e.g., see Curtat (1996) and Amir (2005)), (iii) equilibrium models of stochastic growth without commitment (e.g., Majumdar and Sundaram (1991), Dutta and Sundaram (1992), Amir (1996) or Balbus and Nowak (2004, 2008) for examples of the classic fish war problem or stochastic altruistic growth models with limited commitment between successor generations), (iv) dynamic oligopoly models (see

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[1] For example, see Raghavan, Ferguson, Parthasarathy, and Vrieez (1991) or Neyman and Sorin (2003) for an extensive survey of results, along with references.

Rosenthal (1982), Cabral and Riordan (1994) or Pakes and Ericson (1998)), (v) dynamic negotiations with status quo (see Duggan and Kalandrakis (2007)), (vi) international lending and sovereign debt (Atkeson, 1991), (vii) optimal Ramsey taxation (Phelan and Stacchetti, 2001), (viii) models of savings and asset prices with hyperbolic discounting (Harris and Laibson, 2001), among others.[2]

Additionally, in the literature pertaining to economic applications of stochastic games, the central concerns have been broader than the mere question of weakening conditions for the existence of subgame perfect or Markovian equilibrium. Rather, researchers have become progressively more concerned with characterizing the properties of computational implementations, so they can study the quantitative (as well as qualitative) properties of particular subclasses of subgame perfect equilibrium. For example, often one seeks to simulate approximate equilibria in order to assess the quantitative importance of dynamic/time consistency problems (as is done, for example, for calibration exercises in applied macroeconomics). In other cases, one seeks to estimate the deep structural parameters of the game (as, for instance, in the recent work in empirical industrial organization). In either situation, one needs to relate theory to numerical implementation, which requires both (i) sharp characterizations of the set of SMNE being computed, and (ii) constructive fixed point methods that can be tied directly to approximation schemes. Of course, for finite games[3], the question of existence and computation of SMNE has been essentially resolved.[4] Unfortunately, for infinite games, although the equilibrium existence question has received a great deal of attention, results that provide characterization the SMNE set are needed (e.g. to obtain results on the accuracy of approximation methods). Similarly, we still miss results that establish classes of robust equilibrium comparative statics on the space of games which is needed to develop a collection of computable equilibrium comparative statics[5].

The aim of this paper is to address all of these issues in a single unified methodological approach, that being *constructive monotone* method, where our notions of monotonicity include both set inclusion and pointwise partial orders (on spaces of values or pure strategies). Specifically, we study existence, computation, and equilibrium comparatives statics relative to the set of (S)MNE for an important class of stochastic games, namely those with strategic (within period) strategic complementarities[6] and positive externalities (as analyzed by Amir, Curtat or Nowak).

We first prove existence of a *Markovian* NE via strategic dynamic programming methods similar to that proposed in the seminal work of Abreu, Pearce, and Stacchetti (1990) (henceforth, APS).[7] We refer to this as a more "indirect" method, as we focus exclusively on equilibrium values (rather than characterizing the set of strategies that implement those equilibrium values). Our method differs from those of the traditional APS literature in at least two directions. Perhaps most importantly, we study the existence of short memory or Markovian equilibria, as opposed to broad classes of sequential or subgame perfect equilibria.[8] Additionally, our

[2]Also, for an excellent survey including other economic applications of stochastic games, please see Amir (2003).

[3]By "finite game" we mean a dynamic/stochastic game with a (a) finite horizon and (b) finite state and strategy space game. By an "infinite game", we mean a game where either (c) the horizon is countable (but not finite), or (d) the action/state spaces are a continuum. We shall focus on stochastic games where both (c) and (d) are present.

[4]For example, relative to existence, see Federgruen (1978) and Rieder (1979) (or more recently Chakrabarti (1999)); for computation of equilibrium, see Herings and Peeters (2004); and, finally, for estimation of deep parameters, see Pakes and Pollard (1989); or more recently Aguirregabiria and Mira (2007), Pesendorfer and Schmidt-Dengler (2008), Pakes, Ostrovsky, and Berry (2007) and Doraszelski and Satterthwaite (2010).

[5]There are exceptions to this remark. For example, in Balbus and Nowak (2004), a truncation argument for constructing a SMNE in symmetric games of capital accumulation is proposed. In general, though, in this literature, a unified approach to approximation and existence has not been addressed.

[6]By this we mean supermodularity of a auxiliary game. See Echenique (2004) for a notion of a supermodular extensive form games.

[7]Our results are also related to recursive saddlepoint type methods, which began in the work of Kydland and Prescott (1980), but are also found in the work of Marcet and Marimon (2011) and Messner, Pavoni, and Sleet (2011).

[8]It bears mentioning, we focus on short-memory Markovian equilibrium because this class of equilibrium has

strategic dynamic programming method works directly in function spaces (as opposed to spaces of correspondences), where compatible partial orders and order topologies can be developed to provide a unified framework to study the convergence and continuity of iterative procedures. The latter methodological issue is key (e.g. for uncountable action and state spaces), as it implies we can avoid many of the difficult technical problems associated measurability and numerical implementations using set-approximation techniques.

Next, we strengthen the conditions on the noise of the game, and we propose a different (direct) monotone method for proving existence of *stationary* MNE under different (albeit related) conditions than those used in previous work (e.g., Curtat (1996), Amir (2002, 2005) or Nowak (2007)). In this case, we are able to improve on existence results for SMNE relative to the literature a great deal. First, we are able to consider existence of SMNE in broader spaces of strategies. Second, we give conditions under which the set of SMNE values form a complete lattice of Lipschitz continuous functions. And thirdly, not only are we able to show that SMNE exist, but we are able in principle to compute these SMNE via monotone iterative procedures, and provide conditions under which extremal SMNE can be *uniformly* approximated as the limit of sequences generated by iterations on our fixed point operator.

Finally, unlike the existing work in the literature, our monotone iterative methods apply for the infinite horizon games, as well as finite horizon games. This is particularly important for numerical implementations. Specifically, we are able to provide conditions under which infinite horizon SMNE are the limits of truncated finite horizon stochastic games. We are also able to provide conditions for *monotone comparative statics* results on the set of SMNE for the *infinite horizon game*, as well as describe how equilibrium comparative statics can be computed. This is particularly important, when one seeks to construct a stable selections of the set of SMNE that are numerically (and theoretically) tractable as functions of the deep parameters of the economy.

The rest of the paper is organized as follows. Section 2 states the formal definition of an infinite horizon, stochastic game. Under general conditions, our main result, on Markov equilibrium existence and its value set approximation, can be found in section 3. Then in section 4 we propose a direct method for Markov Stationary Nash equilibrium existence, and computation. In section 4.4, we present related comparative statics and equilibrium dynamics results. Finally in section 5, we discuss possible applications of our results.

2 The Class of Stochastic Games

We consider a n-player discounted infinite horizon stochastic game in discrete time. The primitives of the class of games are given by the tuple $\left\{S, (A_i, \tilde{A}_i, \beta_i, u_i)_{i=1}^n, Q, s_0\right\}$, where $S = [0, \bar{S}] \subset \mathbb{R}^k$ is the state space, $A_i \subset \mathbb{R}^{k_i}$ player i action space with $A = \times_i A_i$, β_i is the discount factor for player i, $u_i : S \times A \to \mathbb{R}$ is the one-period payoff function, and $s_0 \in S$ the initial state of the game. For each $s \in S$, the set of feasible actions for player i is given by $\tilde{A}_i(s)$, which is assumed to be compact Euclidean interval in $\mathbb{R}^{k_i}$. By Q, we denote a transition function that specifies for any current state $s \in S$ and current action $a \in A$, a probability distribution over the realizations of next period states $s' \in S$.

Using this notation, a formal definition of a (Markov, stationary) strategy, payoff, and a Nash equilibrium can now be stated as follows. A *strategy* for a player i is denoted by $\Gamma_i = (\gamma_i^1, \gamma_i^2, \ldots)$, where γ_i^t specifies an action to be taken at stage t as a function of history of all states s^t, as well as actions a^t taken as of stage t of the game. If a strategy depends on a partition of histories limited to the current state s_t, then the resulting strategy is referred to as *Markov*. If for all stages t,

been the focus of a great deal of applied work. Our methods also can be adapted to studying the set of subgame perfect/sequential equilibrium. We should also mention a very interesting papers by Cole and Kocherlakota (2001), and Doraszelski and Escobar (2012) that also pursue a similar ideas of trying to develop APS type procedures in function spaces for Markovian equilibrium (i.e., methods where continuation structures are parameterized by functions).

we have a Markov strategy given as $\gamma_i^t = \gamma_i$, then strategy Γ_i for player i is called a *Markov-stationary strategy*, and denoted simply by γ_i. For a strategy profile $\Gamma = (\Gamma_1, \Gamma_2, \ldots, \Gamma_n)$, and initial state $s_0 \in S$, the expected payoff for player i can be denoted by:

$$U_i(\Gamma, s_0) = (1 - \beta_i) \sum_{t=0}^{\infty} \beta_i^t \int u_i(s_t, a_t) dm_i^t(\Gamma, s_0),$$

where m_i^t is the stage t marginal on A_i of the unique probability distribution (given by Ionescu–Tulcea's theorem) induced on the space of all histories for Γ. A strategy profile $\Gamma^* = (\Gamma_i^*, \Gamma_{-i}^*)$ is a *Nash equilibrium* if and only if Γ^* is feasible, and for any i, and all feasible Γ_i, we have

$$U_i(\Gamma_i^*, \Gamma_{-i}^*, s_0) \geq U_i(\Gamma_i, \Gamma_{-i}^*, s_0).$$

3 MNE via APS methods in Function Spaces

The approach[9] we take in this section to prove existence of MNE and approximate its value set, is the strategic dynamic programming approach based on the seminal work of Abreu, Pearce, and Stacchetti (1986, 1990) adapted to a stochastic game[10]. In the original strategic dynamic programming approach, dynamic incentive constraints are handled using correspondence-based arguments. For each state $s \in S$, we shall envision agents playing a one-shot stage game with the continuation structure parameterized by a correspondence of continuation values, say $v' \in \mathscr{V}(S)$ where $\mathscr{V}(S)$ is the space of nonempty, bounded, upper semincontinuous correspondences (for example). Imposing incentive constraints on deviations of the stage game under some continuation promised utility v', a natural operator B, that is monotone under set inclusion can be defined that transforms $\mathscr{V}(S)$. By iterating on a "value correspondence" operator from a "greatest" element of the collection $\mathscr{V}(S)$, the operator is shown to map down the "largest set" $\mathscr{V}(S)$, and then by appealing to standard "self-generation" arguments, it can be shown a decreasing subchain of subsets can be constructed, whose pointwise limit in the Hausdorff topology is the greatest fixed point of B. Just as in APS for a repeated game, this fixed point turns out to be the set of sustainable values in the game, with a sequential / subgame perfect equilibrium being any measurable selection from the strategy set supporting this limiting correspondence of values.

We now propose a new procedure for constructing all measurable (possibly nonstationary) Markov Nash equilibria for our infinite horizon stochastic game. Our approach[11] is novel, as we operate directly in *function spaces*[12], i.e. a set of bounded measurable functions on S valued in

[9]To motivate our approach in this section, recall a classical result on the existence for a subgame perfect equilibrium in infinite horizon stochastic games first offered by Mertens and Parthasarathy (1987), and later modified, by other authors (e.g., Solan (1998) or Maitra and Sudderth (2007)). See also Harris, Reny, and Robson (1995) for a related argument on subgame perfect equilibria in repeated games with public randomization.

[10]The APS method also is related to the work of Kydland and Prescott (1977, 1980), and now referred to as KP), but is distinct. In the KP approach (and more general "recursive saddlepoint methods" ala Marcet and Marimon (2011) and Messner, Pavoni, and Sleet (2011)), one develops a recursive dual method for computing equilibrium, where Lagrange multipliers become state variables. In this method, the initial condition for the dual variables has to made consistent with the initial conditions of the incentive constrained dynamic program. To solve this last step, Kydland-Prescott propose a correspondence based method that bears a striking resemblance to APS method.

[11]It bears mentioning that for dynamic games with more restrictive shocks spaces (e.g., discrete or countable), KP/APS procedures has been used extensively in economics in recent years: e.g. by Kydland and Prescott (1980), Atkeson (1991), Pearce and Stacchetti (1997), Phelan and Stacchetti (2001) for policy games, and Feng, Miao, Peralta-Alva, and Santos (2009) for dynamic competitive equilibrium in recursive economies. Related methods have been proposed for theoretical work in stochastic games (e.g., Mertens and Parthasarathy (1987) and Chakrabarti (1999, 2008)).

[12]For example, see Phelan and Stacchetti (2001, p. 1500-1501), who discusses such possibility in function spaces. In this section, we show such a procedure is analytically tractable, and discuss it computational advantages in addition.

$\mathbb{R}^n$. Our construction is related to the Cole and Kocherlakota (2001) study of Markov-private information equilibria by the APS type procedure in function spaces. As compared to their study our treats different class of games (with general payoff functions, public signals and uncountable number of states) though. Also, recently and independently of our results Doraszelski and Escobar (2012) establish an APS type procedure in function spaces for Markovian equilibria in repeated games with imperfect monitoring. Again their construction differs from ours as they require a finite number of actions, countable number of states and payoff irrelevant shocks. All of these are necessary in their approach to preserve the measurability of a value function, but not is our case as measurability requirement is easily satisfied for extremal equilibria.

For our arguments in this section, we shall require the following assumptions. We postpone commenting them for the next section.

Assumption 1 *We let:*

- u_i *is continuous on* $S \times A$*,* u_i *is bounded by* 0 *and* $\bar{u}$*,*
- u_i *is supermodular in* a_i *(for any* s, a_{-i}*), and has increasing differences in* $(a_i; a_{-i}, s)$*, and is increasing in* (s, a_{-i})*, (for each* a_i*),*
- *for all* $s \in S$ *the sets* $\tilde{A}_i(s)$ *are compact intervals and multifunction* $\tilde{A}_i(\cdot)$ *is upper hemicontinuous and ascending under both (i) set inclusion i.e. if* $s_1 \leq s_2$ *then* $\tilde{A}_i(s_1) \subseteq \tilde{A}_i(s_2)$*, and (ii) Veinott's strong set order* $\leq_v$ *(i.e.,* $\tilde{A}_i(s_1) \leq_v \tilde{A}_i(s_2)$ *if for all* $a_{1i} \in \tilde{A}_i(s_1), a_{2i} \in \tilde{A}_i(s_2)$*,* $a_{1i} \wedge a_{2i} \in \tilde{A}_i(s_1)$ *and* $a_{1i} \vee a_{2i} \in \tilde{A}_i(s_2)$*)*
- $Q(ds'|\cdot,\cdot)$ *has Feller property on* $S \times A$[13]*,*
- $Q(ds'|s,a)$ *is stochastically supermodular in* a_i *(for any* s, a_{-i}*), has stochastically increasing differences in* $(a_i; a_{-i}, s)$*, and is stochastically increasing with* a, s*,*
- $Q(ds'|s,a)$ *has density* $q(s'|s,a)$ *with respect to some* σ *finite measure* μ *i.e.* $Q(A|s,a) = \int_A q(s'|s,a)\mu(ds')$*. Assume that* $q(s'|s,\cdot)$ *is continuous and bounded for all* (s', s) *and for all* $s \in S$
$$\int_S ||q(s'|s,\cdot)||_\infty \mu(ds') < \infty,$$
- *support of* Q *is independent of* a*.*

We know describe formally our APS type method. Let V be the space of bounded, increasing[14], measurable value functions on S with values in $\mathbb{R}^n$, and $\mathcal{V}$ the set of all subsets of V partially ordered by set inclusion. Define an auxiliary (or, super) one period n-player game $G_v^s = (\{1,\ldots,n\}, \{\tilde{A}_i(s), \Pi_i\}_{i=1}^n)$, where payoffs depend on a weighted average of (i) current within period payoffs, and (ii) a vector of expected continuation values $v \in V$, with weights given by a discount factor:

$$\Pi_i(v_i, s, a) := (1-\beta_i)u_i(s,a) + \beta_i \int_S v_i(s')Q(ds'|s,a),$$

where $v = (v_1, v_2, \ldots, v_n)$, and the state $s \in S$. As $v \in V$ is increasing function, under our assumptions, G_v^s is a supermodular game. Therefore, G_v^s has a nonempty complete lattice of pure strategy Nash equilibria $NE(v,s)$ (e.g., see Topkis (1979) or Veinott (1992)) and corresponding values $\Pi(v,s)$. Having that, for any subset of functions $W \in \mathcal{V}$, define the operator B to be

$$B(W) = \bigcup_{v \in W} \{w \in V \in \mathcal{V} : w(s) = \Pi(v,s,a^*),\ a^* \in NE(v,s)\}.$$

[13]That is $\int_S f(s')Q(ds'|\cdot,\cdot)$ is continuous whenever f is continuous and bounded.

[14]In our paper we use increasing / decreasing terms in their week meaning.

We denote by $V^* \in \mathcal{V}$ the set of equilibrium values corresponding to all monotone Markovian equilibria of our stochastic game. It is immediate that B is both (i) increasing under set inclusion, and (ii) transforms the space $\mathcal{V}$ into itself. This fact implies, among other things, that $B : \mathcal{V} \to \mathcal{V}$ has the greatest fixed point by Tarski's theorem (as $\mathcal{V}$ ordered under set inclusion is a complete lattice). Now, as opposed to traditional APS, where sequential/subgame perfect equilibrium are the focal point of the analysis, it turns out the greatest fixed point of our operator B will generate the set of *all* (monotone, possibly nonstationary) Markovian equilibria values V^* in the function space $\mathcal{V}$. Finally, we offer iterative methods to compute the set of MNE values in the game. Consider a sequence of subsets of (n-tuple) equilibrium value functions (on S) $\{W_t\}_{t=1}^{\infty}$ generated by iterations on our operator B from the initial subset $W_1 = V$ (with $W_{t+1} = B(W_t)$). By construction, $W_1 = V$ is mapped down (under set inclusion by B) and a subchain of decreasing subsets $\{W_t\}$ converge to V^*, the greatest fixed point of B.

We are now ready to summarize these results in the next theorem.

Theorem 3.1 *Assume 1. Then:*

1. *operator B has the greatest fixed point $W^* \neq \emptyset$ with $W^* = \lim_{t\to\infty} W_t = \bigcap_{t=1}^{\infty} W_t$,*

2. *We have $V^* = W^*$.*

The above theorem establishes among others that, the stochastic game has a (possibly nonstationary) Markov Nash Equilibrium in monotone strategies on a minimal state space of current state variables. Observe that conditions to establish that fact are weaker than one imposed by Curtat (1996) or Amir (2002). Specifically we do not require smoothness of the primitives nor any diagonal dominance conditions that assure that the auxiliary game has a unique Nash equilibrium, that is moreover continuous with the continuation value. It is also important to note that our methods allow for a more general state space than many related papers in the existing literature (e.g., Amir (2002, 2005) where the state space is often assumed to be one-dimensional). Seven other comments are now in order.

1. Theorem 3.1.1 follows from a standard application of the Tarski-Kantorovich theorem (e.g., Dugundji and Granas (1982), Theorem 4.2, p. 15). In our application of this theorem, as order convergence and topological convergence coincide in our setting, the lower envelope of the subchains generated by $\{W_t\}$ under set inclusion is equal to the topological limit of the sequence, which greatly simplifies computations in principle.

2. For any strategic dynamic programming argument (e.g., traditional APS in spaces of correspondences or our method in function spaces), for the method to make sense, it requires the auxiliary game to have a Nash equilibrium. In our situation we assume the auxiliary game is a game of strategic complementarity. Of course, alternative topological conditions could be imposed but measurability of the value function has to be carefully analyzed. In our paper, for an easy comparison with the result of monotone operator method presented in the next section, we keep supermodular game specification.

3. The assumption of the full (or invariant) support is critical for this "recursive type" method to work. That is, by having a Markov selection of values from V^*, we can generate supporting Markov (time and state dependent) Nash equilibrium.

4. Observe that the procedure does not imply that the (Bellman type) equation $B(V^*) = V^*$ is satisfied for a *particular* value function v^*; rather, only by a *set* of value functions V^*. Hence, generally existence of a stationary Markov Nash equilibrium cannot be deduced using these arguments. Also, our method is in contrast to the original APS method, where for any $w(s)$, one finds a continuation v; hence, the construction for that method becomes *pointwise*, and for any state $s \in S$, one can select a different continuation function v. This implies the equilibrium strategies that sustain any given point in the equilibrium value set are not only state dependent, but also continuation dependent. In our method this is not the case. This difference, in principle, could have significant implications for computation.

5. On a related matter, the set of Markov Nash Equilibrium values is generally a subset of the APS (sequential / subgame perfect equilibrium) value set. One should keep in mind, for repeated games, the two sets need not be very different[15].

6. The characterization of a sequential NE strategy obtained using standard APS method in general is very weak (i.e., all one can conclude, is that there exists some measurable value function that are selections from V^*). In our case, as we focus on an operator B, that maps in space of *functions,* we resolve this selection issue at the stage of defining the operator.

7. Finally, Judd, Yeltekin, and Conklin (2003) and Sleet and Yeltekin (2003) offer a technique for APS set approximation. For this reason they must *convexify* the set of continuation values. This step usually involves introducing sunspots or correlations devices into the game. Similar computation method can be applied in our context as well, but is substantially simpler at leats for the four reasons: (i) we do not need any of this convexification or correlation, as our method delivers two extremal MNE, (ii) we define equilibria on a minimal state space, (iii) we analyze equilibria that are time/state dependent only, and moreover (iv) we operate directly in function spaces. The last point needs more attention. From the first glance the traditional APS computation procedure is easier as the computation can be conducted pointwise, i.e. for any (of a finite number) state separately, while our not. However, as our withinperiod game is supermodular we can easily compute the extremal withinperiod NE by simple monotone iterations (even for an uncountable no. of states) by e.g. projection methods on the space of polynomials. Moreover and most importantly, we establish (in the next section) an important equilibrium approximation result for two of extremal MSN *equilibria directly.*

4 SMNE via Monotone Operators

The aim of this section is to both (i) prove the existence of a *stationary* Markov Nash Equilibrium (SMNE), and (ii) provide a simple successive approximation scheme for computing particular elements of this equilibrium set. The methods developed in this section of the paper are essentially value iteration procedures in pointwise partial orders on spaces of bounded value functions that map subsets of functions to subset of functions, incorporating the strategic restrictions imposed by the game implicitly in their definition.What is particularly important is that, as compared to APS methods develop in section 3, we also can characterize (and compute) the set of pure strategies the support particular extremal equilibrium values, and well as discuss pointwise equilibrium comparative statics. We can also show SMNE for infinite horizon games can be computed as the limit of finite horizon stochastic games. Sufficient conditions for any of these results have not been produced in any of the existing literature.

Similarly to section 3) we now analyze the structure of equilibria for the infinite horizon game by appealing to equilibrium in the auxiliary game G_v^s. Remind that, if we denote by $\Pi(v, s)$ a vector of values generated by any Nash equilibrium of G_v^s, then it turns out to show the existence of a stationary Markov equilibrium value in the original discounted infinite horizon stochastic game, it suffices to show existence of a (measurable) "fixed point" selection in this auxiliary game (i.e., a measurable selection $v^*(s) \in \Pi(v^*, s)$).

The existence of such fixed point has typically been achieved in the literature by developing applications of nonconstructive topological fixed point theorems on some nonempty, compact and (locally) convex space of value functions $\mathcal{V}$. For example, using this method, Nowak and Raghavan (1992) are able to show existence of a measurable value v^* of the original stochastic game applying Fan-Glicksberg's generalization of Kakutani fixed point theorem. In particular, they deduce the existence of a correlated equilibrium of a stochastic game using a measurable selection theorem due to Himmelberg. This method has been further developed by Nowak (2003), where by adding a specific type of stochastic transition structure for the game (e.g., assuming

[15]For example, as Hörner and Olszewski (2009) show, the Folk theorem can hold for repeated games with imperfect monitoring and finite memory strategies. Further, Barlo, Carmona, and Sabourian (2009) show a similar results for repeated games with perfect monitoring, rich action spaces and Markov equilibria.

a particular mixing structure), the existence of a measurable Markov stationary equilibrium is obtained.[16]

When using the auxiliary game approach, note that one can make significant progress, if it is possible to link a continuation value, say v', to a particular equilibrium value[17] $v(\cdot) \in \Pi(v', \cdot)$ from the *same set of values.* There are many approaches in the literature for implementing this idea. One method is to assume that the auxiliary game has a unique equilibrium (and, hence, a unique equilibrium value $\Pi(v, \cdot) \in CM$, where CM is the set of Lipschitz continuous functions on the state space S). This is precisely the approach that has been recently taken by many authors in the literature (e.g., Curtat (1996) and Amir (2002)). Conditions required to apply this argument are strong, requiring restrictive assumptions on the game's primitives involving strong concavity/diagonal dominance of the game payoffs/transitions, Lipschitzian structure for payoffs/transition structure in the auxiliary game, as well as stochastic supermodularity conditions. If these conditions are present, one can show the existence of monotone, Lipschitz continuous stationary equilibrium via e.g., Schauder's theorem.

Another interesting idea for obtaining the needed structure to resolve these existence issues for the original game via the one-shot auxiliary game, is to develop a procedure for selecting from Π an upper semi-continuous, increasing function (i.e., distribution functions) on a compact interval of the *real line* and observing that a set of such functions is weakly compact. This approach was first explored by Majumdar and Sundaram (1991) and Dutta and Sundaram (1992) in the important class of dynamic games (e.g., dynamic resource extraction games). More recently, Amir (2005) has generalized this argument to a class of stochastic supermodular games, where both values and pure strategies for SMNE are shown to exist in spaces of increasing, upper semi-continuous functions. Of course, the most serious limitation of this purely topological approach in the literature is that to date, the authors have been forced to severely restrict the state space of the game, as well as requiring a great deal of complementarity between actions and states (i.e., assumptions consistent with the existence of monotone Markov equilibrium), to keep the topological argument tractable. Perhaps equally as important is the fact that this method loses all hope of preserving monotone comparative statics results in the supermodular game (in deep parameters).

In our approach, we will also follow this sort of analysis using the auxiliary game, but we show how under our assumptions, we are able to define a *single valued* map, than maps between general spaces of values functions, without imposing uniqueness of Nash equilibrium in the one-shot game, nor requiring monotone SMNE. As will be clear in the sequel, our key point of departure from some of this existing literature is that we adopt the specific noise structure. Although this noise structure has been already used a great deal in the existing literature (e.g. in (Nowak and Szajowski, 2003), (Balbus and Nowak, 2004, 2008) or (Nowak, 2007)), its full power has not been explored so far in the direction of computation and computable equilibrium comparative statics. So using this assumption, we are able to characterize appropriate monotone operators in spaces of values and pure strategies that preserves complementarities between the periods (and onto the infinite horizon game).

4.1 Assumptions for MSNE computation

We first state some initial conditions on the primitives of the game that are required for our methods in this section.

Assumption 2 (Preferences) *For $i = 1, \ldots, n$ let:*

- *u_i be continuous on A and measurable on S, with* $0 \leq u_i(s, a) \leq \overline{u}$*,*

[16]See also papers by Nowak and Szajowski (2003) or Balbus and Nowak (2004, 2008) for a related mixing assumptions on stochastic transition.

[17]That is, one can define a *single-valued* self map $Tv(\cdot) = \tilde{\Pi}(v, \cdot)$ on a nonempty, convex and compact function space, where $\tilde{\Pi}$ is some selection from Π.

- *$(\forall a \in A)$ $u_i(0, a) = 0$,*
- *u_i be increasing in a_{-i},*
- *u_i be supermodular in a_i for each (a_{-i}, s), and has increasing differences in $(a_i; a_{-i})$,*
- *for all $s \in S$ the sets $\tilde{A}_i(s)$ are nonempty, compact intervals and $s \to \tilde{A}_i(s)$ is a measurable correspondence.*

Assumption 3 (Transition) *Let Q be given by:*

- *$Q(\cdot|s, a) = g_0(s, a)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a)\lambda_j(\cdot|s)$, where*
- *for $j = 1, ..., L$ the function $g_j : S \times A \to [0, 1]$ is continuous on A and measurable on S, increasing and supermodular in a for fixed s, and $g_j(0, a) = 0$ (clearly $\sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1$),*
- *$(\forall s \in S, j = 1, \ldots, L)$ $\lambda_j(\cdot|s)$ is a Borel transition probability on S,*
- *$(\forall j = 1, \ldots, L)$ function $s \to \int_S v(s')\lambda_j(ds'|s)$ is measurable and bounded for any measurable and bounded (by some predefined constants) v,*
- *δ_0 is a probability measure concentrated at point 0.*

Although assumptions here are related to those made in recent work by Amir (2005) and Nowak (2007), there does exist some important differences. Before discussing them (in section 4.2) we state our main result.

4.2 Existence and Computation of SMNE

We now discuss existence and computation of SMNE. Let $Bor(S, \mathbb{R}^n)$ to be the set of Borel measurable functions from S into $\mathbb{R}^n$, and consider the subset:

$$\mathcal{B}^n(S) := \{v \in Bor(S, \mathbb{R}^n) : \forall_i v_i(0) = 0, ||v_i|| \leq \bar{u}\}.$$

Equip the space $\mathcal{B}^n(S)$ with a pointwise partial product order. For a vector of continuation values $v = (v_1, v_2, \ldots, v_n) \in \mathcal{B}^n(S)$, we can now analyze the auxiliary one-period, n-player game G_v^s with action sets $\tilde{A}_i(s)$ and payoffs:

$$\Pi_i(v_i, s, a_i, a_{-i}) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, a_{-i}) \int_S v_i(s')\lambda_j(ds'|s).$$

Under assumptions 2 and 3, this auxiliary game G_v^s is a supermodular game for any (v, s), hence it possesses a greatest $\overline{a}(s, v)$ and least $\underline{a}(s, v)$ pure strategy Nash equilibrium (see, Topkis (1979) and Vives (1990)), as well as corresponding greatest $\overline{\Pi}^*(v, s)$ and least $\underline{\Pi}^*(v, s)$ equilibrium values, where $\Pi^*(v, s) = (\Pi_1^*(v, s), \Pi_2^*(v, s), \ldots, \Pi_n^*(v, s))$.

We are now prepared to state the first main theorem of this section concerning the existence of SMNE. Further, we verify existence constructively by providing an explicit successive approximation method whose iteration construct SMNE pure strategies *and* values.[18] To do this, we define a pair of extremal value operators $\overline{T}(v)(s) = \overline{\Pi}^*(v, s)$ and $\underline{T}(v)(s) = \underline{\Pi}^*(v, s)$, as well as

[18] One key aspect of our work relative to APS strategic dynamic programming arguments is that our methods construct selections for *both* values and strategies. When using APS strategic dynamic programming methods, even in the simplest stochastic games (e.g., cases of games where measurability is not an issue), constructive methods are only available for the set of sustainable values in any state (not the strategies which support them).

$T^t(v)$, which denote the t-iteration/orbit of the operator $T(v)$ from v. We then recursively[19] generate a sequence of lower (resp., upper) bounds for equilibrium values $\{v^t\}_{t=0}^{\infty}$ (resp., $\{w^t\}_{t=0}^{\infty}$) where $v^{t+1} = \underline{T}(v^t)$ for $t \geq 1$ from the initial guess $v^0(s) = (0, 0, \ldots, 0)$ (resp., $w^{t+1} = \overline{T}(w^t)$ from initial guess $w^0(s) = (\bar{u}, \bar{u}, \ldots, \bar{u})$) for $s > 0$ and $w^0(0) = 0$). For both lower (resp, upper) value iterations, we can also associate sequences of pure strategy Nash equilibrium strategies $\{\phi^t\}_{t=0}^{\infty}$ (resp., $\{\psi^t\}_{t=0}^{\infty}$), with the operators related by $\phi^t = \underline{a}(s, v^t)$ (resp., $\psi^t = \overline{a}(s, w^t)$). With this, our main existence theorem in this section is the following:

Theorem 4.1 (The successive approximation of SMNE) *Under assumptions 2 and 3 we have*

1. *(for fixed $s \in S$) $\phi^t(s)$ and $v^t(s)$ are increasing sequences and $\psi^t(s)$ and $w^t(s)$ are decreasing sequences,*

2. *for all t we have $\phi^t \leq \psi^t$ and $v^t \leq w^t$ (pointwise),*

3. *the following limits exists: $(\forall s \in S)\ \lim_{t\to\infty} \phi^t(s) = \phi^*(s)$ and $(\forall s \in S)\ \lim_{t\to\infty} \psi^t(s) = \psi^*(s)$,*

4. *the following limits exists $(\forall s \in S)\ \lim_{t\to\infty} v^t(s) = v^*(s)$ and $(\forall s \in S)\ \lim_{t\to\infty} w^t(s) = w^*(s)$,*

5. *ϕ^* and ψ^* are stationary Markov Nash equilibria in the infinite horizon stochastic game. Moreover, v^* and w^* are equilibria payoffs associated with ϕ^* and ψ^* respectively.*

Importantly, we now give pointwise bounds for successive approximations relative to any SMNE using upper and lower iterations built from the construction in Theorem 4.1. This result implies a notion of "pointwise bounds" stated as follows[20].

Theorem 4.2 (Pointwise equilibrium bounds for SMNE) *Let assumptions 2 and 3 be satisfied and γ^* be an arbitrary stationary Markov Nash equilibrium. Then $(\forall s \in S)$ we have the equilibrium bounds $\phi^*(s) \leq \gamma^*(s) \leq \psi^*(s)$. Further, if ω^* is equilibrium payoff associated with any stationary Markov Nash equilibrium γ^*, then $(\forall s \in S)$ we have the bounds: $v^*(s) \leq \omega^*(s) \leq w^*(s)$.*

Before we continue, we make a few remarks on how our Theorems 4.1 and 4.2 relate to the known results from section 3. The above result proves the existence of SMNE, while theorem 3.1 in the APS section of the paper applies only to MNE (i.e., computes all the MNE, stationary or not). Moreover the SMNE from theorem 4.1 are not necessarily monotone, like in theorem 3.1 Finally, the theorem establishes existence of extremal SMNE in *pointwise partial orders,* as opposed to set inclusion orders for MNE via APS type methods.

The existence result in Theorem 4.1 is obtained under different assumptions than Curtat (1996), Amir (2002, 2005) or Nowak (2007) for stochastic, supermodular games. Concerning detailed comparisons we proceed in six points.

1. Relative to Curtat and Amir (see (Amir, 2002)), we do not require the payoffs or the transition probabilities to be smooth (e.g., twice continuously differentiable on an open set containing $S \times A$). Additionally, we weaken the implied assumption of Lipschitz continuity of the functions u_i and g_j that is used in both those papers, which appear to be very strong relative to many economic applications[21]. Perhaps most importantly, we also do not impose conditions on payoffs and stochastic transitions that imply "double increasing differences" in the sense of Granot and Veinott (1985) or diagonal dominance conditions (needed for equilibrium uniqueness

[19] Lemmas 7.5, 7.6 and 7.7 show that both $\overline{T}$ and $\underline{T}$ are well defined transformations of $\mathcal{B}^n(S)$.

[20] Of course, if we sup the pointwise bounds across our compact state space, we can get an estimate of uniform bounds also. The point is our iterations are converging in a weaker sense, i.e., in the product topology

[21] E.g. as it rules out payoffs that are consistent with Inada type assumptions, which, for example, are also ruled out in the work of Horst (2005).

in the stage game) for each player in actions and states. Finally, we do not assume *any* increasing differences between actions and states.

2. On the other hand, we do impose a very important condition on the stochastic transitions of the game that is stronger than needed for existence in other papers and section 3. In particular, we assume the transition structure induced by Q can be represented as a convex combination of $L+1$ probability measures, of which one measure is a delta Dirac concentrated at 0. As a result, with probability g_0, we set the next period state to zero; with probability g_j, the distribution is drawn from the nondegenerate distribution λ_j (where, in this latter case, this distribution does not depend on the vector of actions a, but is allowed to depend on the current state s). Also, although we assume each λ_j is stochastically ordered relative to the Dirac delta δ_0, we do not impose stochastic orderings among the various measures λ_j.

This "mixing" assumption for transition probabilities has been discussed extensively in the literature. For example, it was mentioned in Amir (1996), while studied systematically for broad classes of stochastic games in Nowak (2003) and (almost everywhere) by Horst (2005). Further, in many applications, it has recently been used to a great advantage in various papers studying dynamic consistency problems. Surprisingly, the main strength of this assumption has not been fully used until the work of Balbus, Reffett, and Woźny (2009) in the context of paternalistic altruism economies, as well as in this present paper, where we integrate this mixing assumption into our class of stochastic supermodular games.

There are, however, important differences between functions g_j that we use in the current paper, and those found in the existing literature. Specifically, as we are building a monotone continuation method in spaces of *value functions*, and not Euler equations (as, for example, in Balbus, Reffett, and Woźny (2009)), we do not require strict monotonicity of g_j. This issue, although somewhat technical, turns out to be important in the nature of the results we obtain in the paper (e.g., compare with Balbus, Reffett, and Woźny (2009), where it is shown that weak monotonicity of g_j is needed to avoid trivial invariant distribution on bounded state spaces).

3. The assumptions in the paper by Amir (2005) are also related to ours. Our requirements on payoffs and action sets are substantially weaker (i.e., the feasible action correspondences $\widetilde{A}_i(s)$, and payoff/transition structures u_i and g_j are only required to be measurable with s, as opposed to upper semicontinuous as in Amir (2005)). Further, we also eliminate Amir's assumption on increasing differences between players actions and the state variable s, for existence. That is, for *existence,* we do not require *monotone* Markov equilibrium. Additionally, we do not require the class of games to have a *single* dimensional state space (as required for Amir's existence argument for the infinite horizon version of his game).

There is a critical difference between our work and Amir's relative to the specification of a transition Q, and the comparisons here are more subtle. On the one hand, we require these aforementioned mixing conditions for the stochastic transitions for the game to make the infinite horizon game tractable. On the other hand, Amir for the infinite horizon game requires a strong stochastic equicontinuity conditions for the distribution function Q relative to the actions a, which is critical for his existence argument. We do not need this latter assumption. Also, as we do not require any form of continuity of λ_j with respect to the state s, we are not satisfying Amir's assumption T1. Further, although we both require that the stochastic transition structure Q is stochastically supermodular with a, we do not require increasing differences with (a, s) as Amir does, nor stochastic monotonicity of Q in s. Therefore, the two sets of assumptions are incomparable (i.e., one set does not imply the other).

With these changes of assumptions, our results are substantially stronger than Amir's. In particular, aside from verifying the existence of SMNE, we provide methods for constructing them, and in a moment (see section 4.4), we will show how we can obtain equilibrium monotone comparative statics for the infinite horizon game.

More specifically, to better understand the differences in our approach with Amir's, first observe Amir's existence proof is based on a Schauder fixed point theorem. For this, he needs stochastic equicontinuity conditions on the noise to get (weak*) continuity of a best response operator an a compact and convex set of monotone, upper semicontinuous strategies defined on the real line. In contrast, we just construct a sequence[22] of functions whose limit is a (fixed-point) value leading to SMNE. Our method is completely constructive. Therefore, we do not need to require continuity of a best-response operator, nor compactness or convexity of a particular strategy space.

4. It bears mentioning that our assumptions are weaker than those studied in Nowak (2007).

5. Our limiting arguments are based on the topology of *pointwise* convergence[23]. This allows us to state equivalently all our results using fixed point theorems in $\sigma-$complete posets (e.g., see Dugundji and Granas (1982), Theorem 4.1-4.2), where continuity and convergence is always characterized in terms of order topologies. It is precisely here, where our mixing assumption on the noise has its bite, as this leads to a form of monotonicity that is *preserved* via the value function operator to the infinite horizon. For example, in Curtat (1996), one can only manage to show monotonicity of an operator T in gradient orders (i.e., in ∂v, where ∂ denote a vector of (almost everywhere defined) superdifferentials of v). Obviously for such superdifferentials to be guaranteed to be well-behaved, one needs concavity of the value function in equilibrium. Under our mixing assumptions, we are not limited to such cases.

6. It is important to keep in mind our methods obviously apply for finite horizon games. Of course, for such games, our existence and comparative statics results can be obtained under weaker conditions; but for making arguments that stationary MNE are limits of finite horizon games, our assumptions here will be critical.

4.3 Uniform Error Bounds for Lipschitz continuous SMNE

We now turn to error bounds for approximate solutions. This is something that has not been addressed in the current literature. We initially give two motivations for our results in this section. First, observe that the limits provided by theorems 4.1 and 4.2 are only *pointwise*. With slightly stronger assumptions, we can obtain uniform convergence to a set of stationary Markov Nash equilibria. Second, to obtain uniform error bounds, we need make some stronger assumptions on the primitives that allow us to address the question of Lipschitz continuity of equilibrium strategies. Such assumptions are common in applications (compare Curtat (1996) and Amir (2002)).

In this section we assume that S is endowed with a *taxi-norm* $|| \cdot ||_1$[24]. The spaces A_i and A are endowed with a natural maximum-norm. Each function $f : S \to A$ is said to be *M-Lipschitz continuous* if and only if, for all $i = 1, \ldots, n$ $||f_i(x) - f_i(y)|| \leq M||x - y||_1$. Note, if f_i is differentiable, then M-Lipschitz continuity is equivalent to that each partial derivative being bounded above by M.

To obtain corresponding uniform convergence and uniform approximation results we need additional assumptions.

Assumption 4 *For all i, j:*

- *u_i, g_j are twice continuously differentiable on an open set containing*[25] $S \times A$*,*
- *u_i is increasing in (s, a_{-i}) and satisfies cardinal complementarity*[26]*, in a_i and (a_{-i}, s)*

[22] Compare with Nowak and Szajowski (2003) lemma 5, where a related limiting Nash equilibrium result for two player game is obtained.

[23] Which for pointwise partial orders on spaces of values and strategies coincides with *order convergence* in the interval topology. See Aliprantis and Border (1999) lemma 7.16 as applied on our context.

[24] Taxi-norm of vector $x = (x_1, \ldots, x_k)$ is defined as $||x||_1 = \sum_{i=1}^{k} |x_i|$.

[25] Note that this implies that u_i and g_j are bounded Lipshitz continuous functions on compact $S \times A$.

[26] That is, the payoffs are supermodular in a_i and has increasing differences in (a_i, s) and in (a_i, a_{-i}).

- *u_i satisfy a strict dominant diagonal condition in a_i, a_{-i} for fixed $s \in S, s > 0$, i.e. if we denote $a_i \in \mathbb{R}^{k_i}$ as $a_i := (a_i^1, ..., a_i^{k_i})$ then*

$$\forall_{i=1,\dots,n}\forall_{j=1,\dots,k_i} \sum_{\alpha=1}^{n}\sum_{\beta=1}^{k_\alpha} \frac{\partial^2 u_i}{\partial a_i^j \partial a_\alpha^\beta} < 0,$$

- *g_j is increasing in (s,a) and has cardinal complementarity in a_i and s, a_{-i},*
- *g_j satisfy a strict dominant diagonal condition in a_i, a_{-i} for fixed $s \in S, s > 0$,*
- *for each increasing, Lipschitz and bounded by $\overline{u}$ function f, the function $\eta_j^f(s) := \int_S f(s')\lambda_j(ds'|s)$ is increasing and Lipshitz-continuous[27] with a constant $\bar{\eta}$,*
- *$\tilde{A}_i(s) := [0, \tilde{a}_i(s)]$ and each function $s \to \tilde{a}_i(s)$ is Lipschitz continuous and isotone function[28].*

Now, define a set CM^N of N-tuples, increasing Lipschitz continuous (with some constant M and natural number N) functions on S. Observe that CM^N is a complete lattice when endowed with a partial order (and also convex and compact in the sup norm). CM^N is also closely related to the space where equilibrium is constructed in Curtat (1996) and Amir (2002). There are two key differences between our game and that of both authors. First, we allow the choice set A_i to depend on s (where, these authors assume A_i independent of s). Second, under our assumptions, the auxiliary game G_v^0 has a continuum of Nash equilibria, and hence we need to close the Nash equilibrium correspondence in state 0. These technical differences are addressed in lemma 7.8 and proof of the next theorem. Before that we provide a few definitions. For each twice continuously differentiable function $f : A \to \mathbb{R}$ we define:

$$\mathcal{L}(f) := -\sum_{\alpha=1}^{n}\sum_{\beta=1}^{k_\alpha} \frac{\partial^2 f}{\partial a_i^j \partial a_\alpha^\beta}, \qquad U_{i,j,l}^2 := \sup_{s\in S, a\in \tilde{A}(s)} \frac{\partial^2 u_i}{\partial a_i^j \partial s_l}(s,a),$$

$$G_{i,j}^1 := \sup_{s\in S, a\in \tilde{A}(s)} \sum_{\alpha=1}^{L} \frac{\partial g_\alpha}{\partial a_i^j}(s,a), \quad G_{i,j,l}^2 := \sup_{s\in S, a\in \tilde{A}(s)} \sum_{\alpha=1}^{L} \frac{\partial^2 g_\alpha}{\partial a_i^j \partial s_l}(s,a),$$

$$M_0 := \max\left\{ \frac{(1-\beta_i)U_{i,j,l}^2 + \beta_i\bar{\eta}G_{i,j}^1 + \beta_i\bar{u}G_{i,j,l}^2}{-(1-\beta_i)\mathcal{L}(u_i)} : i = 1,\dots,n, j = 1,\dots,k_i, l = 1,\dots,k \right\}.$$

Theorem 4.3 (Lipschitz continuity) *Let assumptions 2, 3, 4 be satisfied. Assume additionally that each $\tilde{a}_i(\cdot)$ is Lipschitz continuous with a constant less then M_0. Then, stationary Markov Nash equilibria ϕ^*, ψ^* and corresponding values v^*, w^* are all Lipschitz continuous.*

Remark 1 *Note that we could relax assumption of transition probability, such that $\int_S v(s')q(ds'|s,a)$ is smooth, supermodular and satisfy strict diagonal property for all Lipschitz continuous v.*

We can now provide conditions for the uniform approximation of SMNE. The results appeal to a version of Amann's theorem (e.g., Amann (1976), Theorem 6.1) to characterize least and greatest SMNE via successive approximations. Further, as a corollary of Theorem 4.1, we also obtain their associated value functions. For this argument, denote by $\mathbf{0}$ (by $\bar{\mathbf{u}}$ respectively) the n-tuple of function identically equal to 0 ($\bar{u}$ respectively) for all $s > 0$. Observe, as under assumptions 4, the auxiliary game has a unique NE value, we have $\underline{T}(v) = \overline{T}(v) := T(v)$. The result is then stated as follows:

[27] This condition is satisfied if each of measures $\lambda_j(ds'|s)$ has a density $\rho_j(s'|s)$ and the function $s \to \rho_j(s'|s)$ is Lipshitz continuous uniformly in s'.

[28] Each coordinate $\tilde{a}_i^j$ is Lipshitz continuous. Notice, this implies that the feasible actions are Veinott strong set order isotone.

Corollary 4.1 (Uniform approximation of extremal SMNE) *Let assumptions 2, 3 and 4 be satisfied. Then* $\lim_{t\to\infty} ||T^t\mathbf{0}-v^*|| = 0$ *and* $\lim_{t\to\infty} ||T^t\overline{\mathbf{u}}-w^*|| = 0$*, with* $\lim_{t\to\infty} ||\phi^t-\phi^*|| = 0$ *and* $\lim_{t\to\infty} ||\psi^t - \psi^*|| = 0$.

Notice, the above corollary assures that the convergence in theorem 4.1 is *uniform*. We also obtain a stronger characterization of the set of SMNE in this case, namely, the set of SMNE equilibrium value functions form a complete lattice.

Theorem 4.4 (Complete lattice structure of SMNE value set) *Under assumptions 2, 3 and 4, the set of stationary Markov Nash equilibrium values* v^* *in* CM^n *is a nonempty complete lattice.*

The above result provides a further characterization of a SMNE strategies, as well as their corresponding set of equilibrium value functions. From a computational point of view, not only are the extremal values and strategies Lipschitzian (as known from previous work), they can also can be uniformly approximated by a simple algorithm. Also observe that the set of SMNE in $CM^{\sum_i k_i}$ is not necessarily a complete lattice.

4.4 Monotone Comparative Dynamics

In the next section, we provide conditions under which our games exhibit equilibrium monotone comparative statics of extremal fixed point values v^*, w^*, as well as the corresponding extremal equilibria ϕ^*, ψ^*. We also prove a theorem on ordered equilibrium stochastic dynamics. With this equilibrium comparative statics question in mind, consider a parametrization of our stochastic game by a set of parameters $\theta \in \Theta$, where Θ is a partially ordered set. We can view θ as representing the deep parameters of the space of games, with its elements including parameters for (i) period preferences u_i, (ii) the stochastic transitions g_j and λ_j, and (iii) feasibility correspondence A_i. Alternatively, we can think of elements θ as being policy parameters of the environment governing the setting of taxes or subsidies (as, for example, in a dynamic policy game with strategic complementarities). Along those lines, we provide parameterized versions of Assumptions 2 and 3 as follows:

Assumption 5 (Parameterized preferences) *For* $i = 1, \ldots, n$ *let:*

- $u_i : S \times A \times \Theta \to \mathbb{R}$ *be a function and* $u_i(\cdot, s, \theta)$ *continuous on* A *for any* $s \in S, \theta \in \Theta$ *with* $u_i(\cdot) \leq \overline{u}$*, and* $u_i(\cdot, \cdot, \theta)$ *is measurable for all* θ*,*
- $(\forall a \in A, \theta \in \Theta)\, u_i(0, a, \theta) = 0$,
- u_i *be increasing in* (s, a_{-i}, θ),
- u_i *be supermodular in* a_i *for fixed* (a_{-i}, s, θ)*, and has increasing differences in* $(a_i; a_{-i}, s, \theta)$,
- *for all* $s \in S, \theta \in \Theta$*, the sets* $\tilde{A}_i(s, \theta)$ *are nonempty, measurable (for given* θ*), compact intervals and a measurable multifunction that is both ascending in the Veinott's strong set order*[29]*, and expanding under set inclusion*[30] *with* $\tilde{A}_i(0, \theta) = 0$.

Assumption 6 (Parameterized transition) *Let* Q *be given by:*

- $Q(\cdot|s, a, \theta) = g_0(s, a, \theta)\delta_0(\cdot) + \sum_{j=1}^{L} g_j(s, a, \theta)\lambda_j(\cdot|s, \theta)$*, where*

[29] That is, $\tilde{A}_i(s, \theta)$ is *ascending in Veinott's strong set order* if for any $(s, \theta) \leq (s', \theta')$, $a_i \in \tilde{A}_i(s, \theta)$ and $a_i' \in \tilde{A}_i(s', \theta') \Longrightarrow a_i \wedge a_i' \in \tilde{A}_i(s, \theta)$ and $a_i \vee a_i' \in \tilde{A}_i(s', \theta')$.

[30] That is, $\tilde{A}_i(s, \theta)$ is *expanding* if $s_1 \leq s_2$ and $\theta_1 \leq \theta_2$ then $\tilde{A}_i(s_1, \theta_1) \subseteq \tilde{A}_i(s_2, \theta_2)$.

- *for* $j = 1,...,L$ *function* $g_j : S \times A \times \Theta \to [0,1]$ *is continuous with a for a given* s, θ*, measurable for given* θ*, increasing in* (s, a, θ)*, supermodular in a for fixed* (s, θ)*, and has increasing differences in* $(a; s, \theta)$ *and* $g_j(0, a, \theta) = 0$ *(clearly* $\sum_{j=1}^{L} g_j(\cdot) + g_0(\cdot) \equiv 1$*),*

- $(\forall s \in S, \theta \in \Theta, j = 1, \ldots, L)$ $\lambda_j(\cdot|s, \theta)$ *is a Borel transition probability on* S*, with each* $\lambda_j(\cdot|s, \theta)$ *stochastically increasing with* θ *and* s*,*

- $(\forall j = 1, \ldots, L)$ *and* $\theta \in \Theta$ *function* $s \to \int_S v(s')\lambda_j(ds'|s, \theta)$ *is measurable for any measurable and bounded* v*,*

- δ_0 *is a probability measure concentrated at point* 0*.*

Notice, in both of these assumptions, we have added increasing difference assumptions between actions and states (as, for example, in Curtat (1996) and Amir (2002)). We first introduce some notation. For a stochastic game evaluated at parameter $\theta \in \Theta$, denote the least and greatest equilibrium values, respectively, as v^*_θ and w^*_θ. Further, for each of these extremal values, denote the associated least and greatest SMNE pure strategies, respectively, as ϕ^*_θ and ψ^*_θ. Our first monotone equilibrium comparative statics theorem is given in the next theorem:

Theorem 4.5 (Monotone equilibrium comparative statics) *Let assumptions 5 and 6 be satisfied. Then, the extremal equilibrium values* $v^*_\theta(s)$*,* $w^*_\theta(s)$ *are increasing on* $S \times \Theta$*. In addition, the associated extremal pure strategy stationary Markov Nash equilibrium* $\phi^*_\theta(s)$ *and* $\psi^*_\theta(s)$ *are increasing on* $S \times \Theta$*.*

In the literature on stochastic games with strategic complementarities, for infinite horizon, we are not aware of any analog result concerning monotone equilibrium comparative statics as in Theorem 4.5. In particular, because of the non-constructive approach to the equilibrium existence problem (that is typically taken in the literature), it is difficult to obtain such a monotone comparative statics without fixed point uniqueness (e.g. Villas-Boas (1997) for the details). Therefore, one key innovation of our approach of the previous section is that for the special case of our games where SMNE are monotone Markov processes, we are able to construct a sequence of parameterized monotone operators whose fixed points are extremal equilibrium selections. As the method is constructive, this also allows us to compute directly the relevant monotone selections from the set of SMNE.

Finally, we state results on dynamics and invariant distribution started from s_0 and governed by a SMNE and transition Q. Before that let us mention that by our assumptions delta Dirac concentrated at 0 is an absorbing state and hence we have a trivial invariant distribution. As a result we do not aim to prove existence of an invariant distribution, but rather characterize a set of all invariant distributions and discuss conditions when it is not a singleton. For this reason let θ be given and by s^f_t denote a process induced by Q and equilibrium strategy f (i.e., $s_0 = s^f$ is an initial value and for $t > 0$), s_{t+1} has a conditional distribution $Q(\cdot|s_t, f(s_t))$. By $\succeq$, we denote the first order stochastic dominance order on the space of probability measures. We have the following theorem:

Theorem 4.6 (Invariant distribution) *Let assumptions 2, 3 and 4 be satisfied.*

- *Then the sets of invariant distributions for processes* $s^{\phi^*}_t$ *and* $s^{\psi^*}_t$ *are chain complete (with both greatest and least elements) with respect to (first-) stochastic order.*

- *Let* $\overline{\eta}(\phi^*)$ *be the greatest invariant distribution with respect to* ϕ^* *and* $\overline{\eta}(\psi^*)$ *the greatest invariant distribution with respect to* ψ^**. If the initial state of* $s^{\phi^*}_t$ *or* $s^{\psi^*}_t$ *is a Dirac delta in* $\bar{S}$*, then* $s^{\phi^*}_t$ *converges weakly to* $\overline{\eta}(\phi^*)$*, and* $s^{\psi^*}_t$ *converges weakly to* $\overline{\eta}(\psi^*)$*, respectively.*

We make a few remarks. First, the above result is stronger than that obtained in a related theorem in Curtat (1996) (e.g., Theorem 5.2). That is, not only we do characterize the set of invariant distributions associated with extremal strategies (which he does not), but we also prove a weak convergence result per the greatest invariant selection. Further, it is worth mentioning if for almost all $s \in S$, we have $\sum_j g_j(s, \cdot) < 1$, we obtain a positive probability of reaching zero (an absorbing state) each period, and hence the only invariant distribution is delta Dirac at zero. Hence, to obtain a *nontrivial* invariant distribution, one has to assume $\sum_j g_j(s, \cdot) = 1$ for all s in some subset of a state space S with positive measure, e.g. interval $[S', \bar{S}] \subset S$ (see Hopenhayn and Prescott (1992), or more recently Kamihigashi and Stachurski (2010)).[31]

Second, Theorems 4.5 and 4.6 also imply results on monotone comparative dynamics (e.g., as defined by Huggett (2003)) with respect to the parameter vector $\theta \in \Theta$ induced by extremal SMNE: ϕ^*, ψ^*. To see this, we define the greatest invariant distribution $\overline{\eta}_\theta(\phi^*_\theta)$ induced by $Q(\cdot|s, \phi^*_\theta, \theta)$, and greatest invariant distribution $\overline{\eta}_\theta(\psi^*_\theta)$ induced by $Q(\cdot|s, \psi^*_\theta, \theta)$, and consider the following corollary:

Corollary 4.2 *Assume 5, 6. Additionally let assumptions of theorem 4.6 be satisfied for all* $\theta \in \Theta$. *Then* $\overline{\eta}_{\theta_2}(\phi^*_{\theta_2}) \succeq \overline{\eta}_{\theta_1}(\phi^*_{\theta_1})$ *as well as* $\overline{\eta}_{\theta_2}(\psi^*_{\theta_2}) \succeq \overline{\eta}_{\theta_1}(\psi^*_{\theta_1})$ *for any* $\theta_2 \geq \theta_1$.

Clearly, the above results points to the importance of having constructive iterative methods for *both* strategies/values, as well as limiting distributions associated with extremal SMNE. Without such monotone iterations, we could not close our monotone comparative statics results. Further, in conclusion, we stress the fact that by weak continuity of operators used to establish invariant distributions, we can also obtain results that lead us to develop methods to *estimate* parameters θ using simulated moments methods (e.g., see Pakes and Pollard (1989), Lee and Ingram (1991), and more recently Aguirregabiria and Mira (2007), for discussion of how this is done, and why it is important).

5 Applications

Applications of our theorems are immediate, and can be used to study the equilibrium in the games of Curtat (1996), Amir (2002), Horst (2005) or Nowak (2007), including examples such as dynamic (price or quantity) oligopolistic competition, stochastic growth models without commitment, problems of dynamic consistency, models with weak social interactions, as well as various dynamic policy games. We now discuss four such applications of our results. We first use our results from section 3 and 4 to prove existence of Markov equilibrium in dynamic oligopoly model. Then we discuss equilibrium dynamics in the dynamic R&D oligopoly model. Third, we show how our methods can be used for analyzing the question of credible government public policies as defined by Stokey (1991). We conclude commenting on generalization of our results per symmetric MSNE of symmetric stochastic games.

5.1 Price competition with durable goods

Consider an economy with n firms competing on customers buying durable goods, that are heterogenous but substitutable to each other. Apart from price of a given good, and vector of competitors goods' prices, demand for any commodity depends on demand parameter s. Each period firms choose their prices, competing a la Bertrand with other's prices. Our aim is to analyze the Markov (Stationary) Nash Equilibrium of such economy. Our paper developed two separate conditions and methods for studying equilibria in such economies, those from section 3 and that from 4. With start with the latter one.

[31] It is also worth mentioning that much of the existing literature does not consider the question of characterizing the existence of *Stationary Markovian equilibrium* (i.e, strategy and invariant distribution pairs).

Payoff of firm i, choosing price $a_i \in [0, \bar{a}]$ is $u_i(s, a_i, a_{-i}, \theta) = a_i D_i(a_i, a_{-i}, s) - C_i(D_i(a_i, a_{-i}, s), \theta)$, where s is a (common) demand parameter, while θ is a cost function parameter. As within period game is Bertrand with heterogenous but substitutable products, naturally the preference assumption 2 is satisfied if demand D_i is increasing with a_{-i}, has increasing differences in (a_i, a_{-i}) and cost function C_i is increasing and convex. As $[0, \bar{a}]$ is single dimensional, u_i is a supermodular function of a_i.

Concerning the interpretation of the assumptions placed on Q in the context of this model: letting $s = 0$ be an absorbing state means that there is a probability that demand will vanish and companies will be driven out of the market. The other assumptions on transition probabilities are also satisfied if $Q(\cdot|s, a) = g_0(s, a)\delta_0(\cdot) + \sum_j g_j(s, a)\lambda_j(\cdot|s)$ and g_j, λ_j satisfy assumptions 3. Interpreting: high prices a today result in high probability for positive demand in the future, as the customer trades-off between exchanging the old product with the new one, and keeping the old product and waiting for lower prices tomorrow. Supermodularity in prices imply that the impact of a price increase on positive demand parameter tomorrow is higher when the others set higher prices. Indeed when the company increases its price today it may lead to a positive demand in the future, if the others have also high prices. But if the others firms set low prices today, then such impact is definitely lower, as some clients may want to purchase the competitors good today instead. Observe, such assumptions guarantee that the stochastic (extensive form) game has the supermodular structure for extremal strategies, the feature that is uncommon for general extensive form games (see Echenique (2004)). That is, if a strategy of a player is increased in the some period $t + \tau$, it leads to a higher value of all players and by our mixing transition assumption increase period t extremal strategies.

The results of the paper (theorem 4.1) prove existence of the greatest and the least Markov stationary Bertrand Equilibrium and allow to compute the equilibria, by a simple iterative procedure. The results extend hence the Curtat (1996) paper example to the non-monotone strategies, characterizing the monopolistic competition economy with substitutable durable goods and varying consumer preferences. Finally, our approximation procedure allow applied researcher to compute and estimate the stochastic properties of these models using the extremal invariant distributions (see theorem 4.6). Finally, if one adds assumptions of theorem 4.5 one obtain monotone comparative statics of the extremal equilibria and invariant distributions (see corollary 4.2), the results absent in the related work.

If the mixing assumption is too restrictive in some applications, we still offer in theorem 3.1 MNE existence, provided assumption 1 is satisfied. Observe that here, as compared to assumptions 2 and 3, we need to add increasing differences between (a_i, s) and monotonicity in s. Such method allow hence to study the monotone equilibria only. The interpretation of such assumption means that high demand today imply high demand in the future, a.o. To justify this Curtat (1996) argues: that "high level of demand today is likely to result in a high level of demand tomorrow because one can assume that not all customers will be served today in the case of high demand." Although, in our paper the MNE existence is obtained under weaker assumption than those of Curtat (1996) or Amir (2002), still the mentioned monotonicity assumption is questionable, as customer rationing is not a part of this game description. Hence, methods developed in section 4 are plausible.

5.2 Dynamic R&D competition

d'Aspremont and Jacquemin (1988) analyze a two stage game between oligopolists choosing the R&D expenditure to reduce costs in the first stage and then compete a la Cournot in the second stage. They analyze the effects of R&D investment spillovers in an (subgame perfect) equilibrium and its optimality.

In the spirit of a pioneering analysis of Ruff (1969) we now extend the R&D competition model to an infinite horizon stochastic game, where each period a two stage game of d'Aspremont and Jacquemin (1988) is played between n oligopolists. For this reason assume that inverse demand is given by $P(Q) = A - bQ$, with $Q = \sum_i q_i$ and production cost functions $c_i = C_i(q_i) = [\overline{S} - s - a_i - \delta \sum_{j \neq i} a_j] q_i$, where $s \in [0, \overline{S}] \subset \mathbb{R}$ is a (drawn each period) common cost parameter (e.g. determined by a business cycle), $\delta \in [0,1]$ is a spillovers parametr and a_i is a investment in an cost reduction R&D process. The cost of a_i units of R&D investment is given by $a_i \to \gamma_i(a_i)$ that is continuous and bounded. Apart from the withinperiod spillovers, higher investment a_i has also intertemporal effects of increasing probabilities of a high cost reduction draw (from a distribution Q) tomorrow.

Every period the profit of an oligopolist (assuming the next stage a Cournot equilibrium is played) is given by

$$\pi_i(s, a_i, a_{-i}) = \frac{1}{b} \left[\frac{A - n(\overline{S} - s - a_i - \delta \sum_{j \neq i} a_j) + \sum_{j \neq i} (\overline{S} - s - a_j - \delta \sum_{k \neq j} a_k)}{n+1} \right]^2 - \gamma_i(a_i).$$

Observe that for a large R&D spillovers $\delta > .5$ the payoff is increasing in a_{-i} (the top-dog strategy effect in dominated by a spillovers effect) and has increasing differences in (a_i, a_{-i}) and (a_i, s). Hence, if only the transition $Q(s, a)$ satisfies our assumption 1 (which is possible if intertemporal investment effects are self reinforcing or independent), the existence and characterization of the whole monotone, Markov equilibrium set is provided by theorem 3.1. If moreover the mixing assumption 3 is satisfied, then existence and computation of MSNE is possible using results of theorem 4.1. For this reason we additionally need to allow $s = 0$ being an absorbing state, but this is justified, e.g. if $\bar{S} \geq A$, i.e. assumption ruling out production possibilities, if the size of the market is too small relative to the unit production cost $\overline{S}$. Of course our results extend to more general payoff functions, decision / state variables, than the specific one analyzed in this example.

5.3 Time-consistent public policy

We now consider a time-consistent policy game as defined by Stokey (1991) and analyzed more recently by Lagunoff (2008). Consider a (stochastic) game between a single household and the government. For any state $k \in S$ (capital level), households choose consumption c and investment i treating level of a government spending G as given. There are no security markets that household can share the risk for tomorrow capital level. The only way to consume tomorrow is to invest in the stochastic technology Q. The within period preferences of household are given by $u(c)$, i.e. household do not obtain utility from public spending G. The government raises revenue by levying flat tax $\tau \in [0,1]$ on capital income, to finance its public spending $G \geq 0$. Each period the government budget is balanced and its within period preferences are given by: $u(c) + J(G)$. The consumption good production technology is given by constant return to scale function $f(k)$ with $f(0) = 0$. The transition technology between states is given by a probability distribution $Q(\cdot | i, k)$, where i denotes household investment. The timing in each period is that the government and household choose their actions simultaneously. Household and government take price R as given which in equilibrium equal $f'(k)$. Assume that u, J, f are increasing, concave and twice continuously differentiable and Q is given by assumption 3.

Household choose:

$$\max_{i \in [0, (1-\tau)Rk]} u((1-\tau)Rk - i) + g(i)\beta \int v_H(s) \lambda(s|k).$$

Observe that objective is supermodular in i and has increasing differences in (i, t), where $t = 1 - \tau$ by the envelope theorem and $-u''(\cdot) \geq 0$. Moreover the objective is increasing in $t = 1 - \tau$ by monotonicity of u.

The government problem is:

$$\max_{t \in [0,1]} u(tRk - i) + g(i)\beta \int v_H(s)\lambda(s|k) + J(Rk(1-t)) + g(i)\beta \int v_G(s)\lambda(s|k).$$

That is, the government maximizes the household utility plus its additional utility from public spending J and its continuation v_G. Again objective is supermodular in $1-\tau$ and has increasing differences in $(t = 1-\tau, i)$ as $-u''(\cdot) \geq 0$. Moreover observe, although the objective is not increasing in i, along the Nash equilibrium of the auxiliary game the objective is increasing in $i^*(v_H)$, again by the envelope theorem.

Although the modes does not fit the general assumptions of the game analyzed in our paper, essentially the same methods as developed in the paper can be applied here to prove existence and compute MSNE. Specifically we can construct an operator on the space of values, that would be monotone (as the within period game is supermodular and the Nash equilibrium of such game is monotone in v_H, v_G).

Some additional interesting points of departure from this above basic specification can also be worked out, including: (i) elastic labor supply choice, or more importantly (ii) adding security markets, investment/insurance firms possessing Q and proving existence of prices decentralizing optimal investment decision i^*. Still observe, however, that here we are able to offer weak assumptions for existence of a stationary credible policy, as well as offer a variety of tools allowing for its constrictive study and computation.

5.4 Symmetric equilibria in symmetric stochastic games

Consider a special case of our stochastic game, where all players have identical preferences $u := u_i$ and action sets $\tilde{A} := \tilde{A}_i \subset \mathbb{R}$. With slight abuse of notation, we denote payoff of a player choosing a_i, when others choose a_{-i} in state s by $u(s, a_i, a_{-i})$. Now observe that for such a special case we can obtain results of theorem 4.1, 4.2 and others from section 4 for *symmetric equilibria* dispensing assumption 2 of increasing differences of u in (a_i, a_{-i}) and supermodularity of g in a in assumption 3. Instead, to guarantee existence of the NE of the auxiliary game we need to add quasi-concavity of g in a, and quasi-concavity of u in a_i.

Indeed, under such additional assumptions the auxiliary game G_v^s has the greatest and the least symmetric Nash equilibrium, both monotone in v by Corollary 2 of Milgrom and Roberts (1994). Hence we can still construct two monotone operators $\overline{T}$,$\underline{T}$ and reconstruct the proofs of theorems 4.1 and 4.2.

Such modification is important as it allows to dispense restrictive assumption of (within period) strategic complementarities between players, but allowing to obtain (between period) strategic complementarities (at least for selected extremal NE values), a necessary feature for our constructive arguments.

An immediate example of this generalization is study of symmetric MSNE in a stochastic version of a private provision of public good game. Let $u(c_i, Y)$ be a payoff from consumption of a private c_i and public good Y. Assume marginal utilities are decreasing and both goods are complements. Endow consumer with income w to be distributed between c_i and private provision y_i. Let a public good be produced using technology $Y = F(\sum_i y_i, s)$, where F is increasing and concave in the first argument. Observe that the function $(y_i, y_{-i}) \to u(w - y_i, F(\sum_j y_j, s))$ does not have increasing differences, but has positive externalities due to free rider problem. Let s parameterize public good stock (i.e. a draw representing a stock from the previous period) or its productivity, while Q represent a process allowing to reduce a future probability of a zero output / productivity, by higher provisions $(y_1, \ldots, y_n)$ today. By theorems 4.1 and 4.2 we can prove existence and approximate the greatest and least symmetric MSNE of such a game.

Finally using this generalization, we can reconsider symmetric MSNE of an Bertrand competition with durable good example (see subsection 5.1) and relax increasing differences assumption of demand D_i with (p_i, p_{-i}) and supermodularity of g and similarly for the symmetric MSNE in the R&D example.

6 Concluding remarks

We have presented two constructive methods for computing MNE in this paper. Under very mild conditions on the game, we are able to develop a strategic dynamic programming approach to our class of stochastic supermodular games, and provide constructive methods for computing the equilibrium value set that is associated with MNE in the game. Aside from allowing us to weaken conditions for existence significantly from those in the existing literature, we are also able to produce strategic dynamic programming methods that focus on *Markovian* equilibrium (as opposed to more general sequential equilibrium). The set of MNE include both stationary and nonstationary Markovian equilibrium, but in all cases, MNE exist on *minimal* state spaces, and our state spaces are allowed to be very general (i.e., compact intervals in finite dimensional Euclidean spaces).

Additionally, under mild mixing assumptions on stochastic transitions, we are able to exploit the complementarity structure of our class of games, and develop very sharp results for iterative procedures on *both* values and pure strategies, which provides very important improvements over strategic dynamic programming/APS type methods that we develop in section 3. For example, under assumptions 1, 2 and 3, we are able to show how to compute both the greatest and least SMNE values $w^*, v^* \in V^* = B(V^*)$ (as, of course, they are also Markov NE), as well as associated extremal pure strategy equilibrium. As for every iteration of our APS operator we can keep track of a greatest and least NE value, clearly, our iterative methods under mixing assumptions verify that for these cases, we have the our APS value set $W_t \subseteq [v_t, w_t]$. So in such case, our iterations in section 4 provide interval bounds on the iterations on our strategic dynamic programming operator. Further, under these mixing assumptions, as the partial orders we use in all cases are chain-complete (i.e., both pointwise and set inclusion orders), we conclude that $V^* \subseteq [v^*, w^*]$. That is, the set of value functions that are associated with MNE belongs to an ordered interval between least and greatest SMNE. So although we cannot show that W_t (for $t \geq 2$) or V^* are ordered intervals of functions, we use our iterative methods to calculate the two bounds using direct techniques of section 4.

This observation leads us to a final important point linking our direct methods with strategic dynamic programming/APS approach. Namely, in Abreu, Pearce, and Stacchetti (1990), they show that under certain assumption any value from $V^*(s)$ can be obtained in a bang-bang equilibrium for a *repeated* game, i.e. one using extremal values from $V^*(s)$. Our direct and constructive methods of section 4 can be hence used to compute two of such extremal values that support equilibrium punishment schemes that actually implement MNE.This greatly sharpens the method by which we support all MNE in our collection of dynamic games. So this allows one to extend some of the ideas that have been developed in the literature on repeated games to settings with state variable, which also helps complement the important work of Judd, Yeltekin, and Conklin (2003) per developing numerical techniques to actually compute the *entire* equilibrium value correspondence. This particular issue will be pursued in our future research.

7 Proofs

7.1 Proof using indirect methods

We first prove a series of important lemmas that we use to prove the central theorem of this section. The main theorem, Theorem 3.1.1, will show that iterations on an operator B starting from some upper element $V \in \mathcal{V}$ that is mapped down (under set inclusion) converge pointwise (in the Hausdorff metric topology on $\mathcal{V}$) to the greatest fixed point W^* of B. Further, in theorem 3.1.2, the greatest fixed point of B under set inclusion, is a set of all Markov Nash Equilibria values in V, i.e. $W^* = V^*$.

Our first lemma considers the fixed point set of G_v^s.

Lemma 7.1 (Complete lattice of $NE(v,s)$) *If $W \in \mathcal{V}$, then $B(W) \cap V \neq \emptyset$.*

Proof of lemma 7.1: Let $v \in W$. Since v is an increasing function, hence $\Pi_i(v_i, s, a)$ is supermodular in a_i and has increasing differences in (a_i, a_{-i}). Therefore G_v^s is a supermodular game parameterized by s. By Theorem 5 in Milgrom and Roberts (1990) it possess the nonempty, complete lattice of pure strategy Nash equilibria. The greatest and least of them are increasing with s by increasing differences assumption between (a_i, s) and monotonicity of $s \rightrightarrows A_i$ in Veinott strong set order (see theorem 6 by Milgrom and Roberts (1990)). Let $s_2 \geq s_1$ and consider the greatest $\overline{a}(s, v)$ (or the least) Nash equilibrium. The following holds for all i:

$$
\begin{aligned}
&\tilde{w}_i(s_2) := \\
&\Pi_i(v_i, s_2, \overline{a}_i(s_2, v_i), \overline{a}_{-i}(s_2, v_i)) = \max_{a_i \in \tilde{A}_i(s_2)} \Pi_i(v_i, s_2, a_i, \overline{a}_{-i}(s_2, v_i)) \geq \\
&\Pi_i(v_i, s_2, \overline{a}_i(s_1, v_i), \overline{a}_{-i}(s_2, v_i)) \geq \Pi_i(v_i, s_1, \overline{a}_i(s_1, v_i), \overline{a}_{-i}(s_2, v_i)) \geq \\
&\Pi_i(v_i, s_1, \overline{a}_i(s_1, v_i), \overline{a}_{-i}(s_1, v_i)) := \tilde{w}_i(s_1).
\end{aligned}
$$

where the first inequality follows as $\tilde{A}_i$ is ascending in the set inclusion order, the second by monotonicity of Π_i with s and the last by monotonicity of $s \to \overline{a}_{-i}(s, v_i)$ and $a_{-i} \to \Pi_i(v_i, s, a_i, a_{-i})$. Therefore, $\tilde{w} = (\tilde{w}_1, ..., \tilde{w}_n)$ is a selection from $B(W) \cap V$. ■

We next prove a result we need to study the convergence properties of B. We use it in the proof of lemma 7.3 and theorem 3.1.

Lemma 7.2 (Convergence of values) *Let $v^t \to v$ in the weak*-topology on $L_\infty(\mu)$,*[32] *and $a^t \to a$ as $t \to \infty$. Then $\Pi(v^t, s, a^t) \to \Pi(v, s, a)$ pointwise in $s \in S$.*

Proof of lemma 7.2: By continuity assumptions 1, we just need to show that:

$$\int_S v_i^t(s')q(s'|s, a^t)\mu(ds') \to \int_S v_i(s')q(s'|s, a)\mu(ds').$$

Indeed, by triangle inequality:

$$
\begin{aligned}
&\left| \int_S v_i^t(s')q(s'|s, a^t)\mu(ds') - \int_S v_i(s')q(s'|s, a)\mu(ds') \right| \\
&\leq \left| \int_S v_i^t(s')q(s'|s, a^t)\mu(ds') - \int_S v_i^t(s')q(s'|s, a)\mu(ds') \right| \\
&+ \left| \int_S v_i^t(s')q(s'|s, a)\mu(ds') - \int_S v_i(s')q(s'|s, a)\mu(ds') \right|.
\end{aligned}
$$

Using bounds on $\bar{u}$ we obtain that the expression above

$$
\begin{aligned}
&\leq \bar{u} \int_S |q(s'|s, a^t) - q(s'|s, a)|\mu(ds') \qquad (1) \\
&+ \left| \int_S v_i^t(s')q(s'|s, a)\mu(ds') - \int_S v_i(s')q(s'|s, a)\mu(ds') \right|.
\end{aligned}
$$

By the definition of weak*-convergence, the second term above converges to 0. We now show that the first term in (1) converges to 0. By Assumption 1, the function under the integral converges

[32] $v_t \to v$ in the weak* topology if and only if, for all $g \in \mathcal{L}_1(\mu)$ $\int_S v_t(s)g(s)\mu(ds) \to \int_S v(s)g(s)\mu(ds)$.

to 0. Note, we have $|q(s'|s,a) - q\left(s'|s,a^t\right) \leq 2||q(s'|s,\cdot)||_\infty$. Since by Assumption 1, $||q(s'|s,\cdot)||_\infty$ is μ integrable function, the convergence of integrals follows from Lebesgue Dominance Theorem.

As a result $\Pi(v^t, s, a^t) \to \Pi(v, s, a)$ pointwise in $s \in S$. ■

Lemma 7.3 (Compactness of W_t) *For each $t \in \mathbb{N}$, W_t is s compact set in the weak*-topology on $L_\infty(\mu)$.*

Proof of lemma 7.3: Since V is a set of functions with the image in $[\mathbf{0}, \bar{\mathbf{u}}]$, by Alaoglu theorem, it is weak*-compact set. We now show that W_t is compact for $t > 1$. To do this, it is sufficient to show $B(W)$ is weak*-compact whenever W is weak*-compact.

Let $w^t \in B(W)$ for all t. By the definition of $B(W)$, we have

$$w^t(s) = \Pi(v^t, s, a^t(s))$$

where $v_t \in W$ and $a^t(s) \in NE(v^t, s)$. As both W and V are compact, without loss of generality, assume $v^t \to v^*$ where $v^* \in W$, and $w^t \to w^*$, where convergence in both cases is with respect to the weak*-topology. Fix $s > 0$. Then, $a^t(s) \to a^*$. We now show that a^* is Nash equilibrium in the reduced game $G^s_{v^*}$. By Lemma 7.2 , we have

$$\Pi(v^t, s, a^t) \to \Pi(v^*, s, a^*)$$

pointwise in s. Hence, a^* is a Nash equilibrium in the static game $G^s_{v^*}$. For each s, we can define $a^*(s)$ as a Nash equilibrium function[33]. Therefore, we have

$$w^*(s) := \Pi(v^*, s, a^*(s))$$

with $w^* \in B(W)$, $a^*(s) \in NE(v^*, s)$. Hence w^* is a weak*-limit of the sequence w_t as it is a pointwise limit. ■

Lemma 7.4 (Self generation) *If $W \subset B(W)$ then $B(W) \subset V^*$.*

Proof of lemma 7.4: Let $w \in B(W)$. Then, we have $v_0(\cdot) := w(\cdot) = \Pi(v_1, \cdot, \sigma^1(\cdot))$ for some $v_1 \in W$ and Nash equilibrium $\sigma^1(s) \in NE(v_1, s)$. Then, since $v_1 \in W$ by the assumption, $v_1 \in B(W)$. Consequently, for $v_t \in W \subset B(W)$ $(t \geq 1)$ we can choose $v_{t+1} \in W$ such that $v_t(\cdot) = \Pi(v_{t+1}, \cdot, \sigma^{t+1}(\cdot))$ and $\sigma^t(\cdot) \in NE(v_{t+1}, \cdot)$. Clearly, the Markovian strategy σ generates payoff vector w. We next need to show this is a Nash equilibrium in the stochastic game for (μ-almost) all initial states. Suppose that only player i uses some other strategy $\tilde{\sigma}_i$. Then, for all t, we have $v^i_t(s) = \Pi_i(v_{t+1}, s, \sigma^t(s)) \geq \Pi_i(v_{t+1}, s, \sigma^{t+1}_{-i}(s), \tilde{\sigma}^{t+1}_i)$. If we take a T truncation[34], $\sigma^{T,\infty} = \left(\left(\tilde{\sigma}^1_i, \sigma^1_{-i}\right), \ldots, \left(\tilde{\sigma}^T_i, \sigma^T_{-i}\right), \sigma^{T+1},\ \sigma^{T+2}, \ldots\right)$, this strategy can not improve a payoff for player i. Indeed,

$$U_i(\sigma, s) \geq U_i(\sigma_{-i}, \sigma^{T,\infty}_i, s) \to U_i(\sigma_{-i}, \tilde{\sigma}_i, s)$$

as $T \to \infty$. This convergence has been obtained as u_i is bounded, and the residum of the sum $U_i(\sigma_{-i}, \sigma^{T,\infty}_i, s)$ depending on $\left(\sigma^{T+1}, \sigma^{T+2}, \ldots\right)$ can be obtained as an expression bounded by $\bar{u}$, multiplied by β^T_i. ■

Proof of theorem 3.1: We prove 1. As $\mathcal{V}$ is a complete lattice, B is increasing, by Tarski theorem, B has the greatest fixed point W^*. Moreover, as B is increasing, $\{W_t\}_{t=0}^\infty$ is a decreasing

[33]Generally a^* is unmeasurable, however for us it is enough to obtain measurability of w^*.

[34]That is agent i uses strategy $\tilde{\sigma}$ up to step T and uses σ after that. The rest agents use σ.

sequence (under set inclusion). Let $V^\infty := \lim_{t\to\infty} W_t = \bigcap_{t=1}^{\infty} W_t$.We need to show that $V^\infty = W^*$.Clearly, $V^\infty \subset W_t$for all $t \in \mathbb{N}$; hence

$$B(V^\infty) = B\left(\bigcap_{t=1}^{\infty} W_t\right) \subset \bigcap_{t=1}^{\infty} B(W_t) = \bigcap_{t=1}^{\infty} W_{t+1} = V^\infty.$$

To show equality, it suffices to show $V^\infty \subset B(V^\infty)$. Let $w \in V^\infty$. Then, $w \in W_t$ for all t. By the definition of W_t and B, we obtain existence of the sequence $v^t \in W_t$ and Nash equilibria a^t such that

$$w(s) = \Pi(v^t, s, a^t).$$

Since V is compact, without loss of generality, assume v^t weakly star converges to v^*. Moreover, $v^* \in V^\infty$, since W_t is a descending set of compact sequences in the weak star topology. Fix arbitrary $s > 0$. Without loss of generality, let $a^t \to a^*$, where a^* is some point from A. We can now show a^* is a Nash equilibrium in the static game $G^s_{v^*}$. Let $a_i \in \tilde{A}_i(s)$. Then, for all $\tau \in \mathbb{N}$:

$$\Pi_i(v_i^\tau, s, a^\tau) \geq \Pi_i(v_i^\tau, s, a^\tau_{-i}, a_i).$$

By lemma 7.2 , if we take a limit in this expression, we obtain a^* a Nash equilibrium in the static game $G^s_{v^*}$, and $w(s) = \Pi(v^*, s, a^*)$. We obtain $w \in B(V^\infty)$. Hence, V^∞ is a fixed point of B, and, by definition $V^\infty \subset W^*$.

To finish the proof, we simply need to show $W^* \subset V^\infty$. Since $W^* \subset V$, $W^* = B(W^*) \subset B(V) = W_1$. By induction, we have $W^* \subset W_t$ for all t; hence, $W^* \subset V^\infty$. Therefore, $W^* = V^\infty$, which completes the proof.

We prove 2. First show that V^* is a fixed point of operator B. Clearly $B(V^*) \subset V^*$. So we just need to show the reverse inclusion. Let $v \in V^*$ and $\sigma = (\sigma_1, \sigma_2, \ldots)$ be a profile supporting v. By assumption 1, $\sigma_{2,\infty} = (\sigma_2, \sigma_3, \ldots)$ must be a Nash equilibrium μ almost everywhere (i.e. a set of initial states S_0 which $\sigma_{2,\infty}$ is not a Markov equilibrium must satisfy $\mu(S_0) = 0$). Define a new profile $\tilde{\sigma}(s) = \sigma_{2,\infty}$ for $s \notin S_0$ and $\tilde{\sigma}(s) = \sigma$ if $s \in S_0$. Let $\tilde{v}$ be equilibrium payoff generated by $\tilde{\sigma}$. Clearly, $\tilde{v} \in V^*$ and is μ measurable and also $v(s) = \Pi(\tilde{v}, s, \sigma_1)$. Thus $v \in B(V^*)$ and hence $V^* \subset B(V^*)$. As a result $B(V^*) = V^*$.

Finally by definition (greatest fixed point) of W^* we conclude that $V^* \subset W^*$. To obtain the reverse inclusion we apply lemma 7.4. Indeed $W^* \subset B(W^*)$, and, therefore, $W^* \subset V^*$ and we obtain that $V^* = W^*$. ■

7.2 Proof using direct methods

We first state three lemmata that prove useful in verifying the existence of SMNE in our game. The lemmata, in addition, is also useful in characterizing monotone iterative procedures for constructing least and greatest SMNE (relative to pointwise partial orders on $\mathcal{B}^n(S)$). More specifically, the lemma concerns the structure of Nash equilibria (and their associated corresponding equilibrium payoffs) in our auxiliary game G^s_v.

Lemma 7.5 (Monotone Nash equilibria in G^s_v) *Under assumptions 2 and 3, for every $s \in S$ and value $v \in \mathcal{B}^n(S)$, the game G^s_v has the maximal Nash equilibrium $\overline{a}(v, s)$, and minimal Nash equilibrium $\underline{a}(v, s)$. Moreover, both equilibria are increasing in v.*

Proof of lemma 7.5: Without loss of generality fix $s > 0$. Define auxiliary one shot game, say $\Delta(\tau)$, with an action space A, and payoff function for player i given as

$$H_i(a, \tau) := (1 - \beta_i)u_i(s, a_i, a_{-i}) + \beta_i \sum_{j=1}^{L} \tau_{i,j} g_j(s, a_i, a_{-i}),$$

where $\tau := [\tau_{i,j}]_{i=1,\ldots,n,j=1,\ldots,L} \in \mathcal{T} := \mathbb{R}^{n\times L}$ is endowed with the natural pointwise order. As supermodularity of a function on a sublattice of a directed product of lattices implies increasing differences (see Topkis (1998) theorem 2.6.1) clearly, for each $\tau \in \mathcal{T}$, the game $\Delta(\tau)$ is supermodular, and satisfies all assumptions of Theorem 5 in Milgrom and Roberts (1990). Hence, there exists a complete lattice of Nash equilibrium, with the greatest Nash equilibria given by $\overline{NE}\Delta(\tau)$, and the least Nash equilibrium given by $\underline{NE}\Delta(\tau)$. Moreover, for arbitrary i, the payoff function $H_i(a,\tau)$ has increasing differences in a_i and τ; hence, $\Delta(\tau)$ also satisfies conditions of Theorem 6 in Milgrom and Roberts (1990). As a result, both $\overline{NE}\Delta(\tau)$ and $\underline{NE}\Delta(\tau)$ are increasing in τ.

Step 2: For each $s \in S$, the game G_v^s is a special case of $\Delta(\tau)$ where $\tau_{i,j} = \int_S v_i(s')\lambda_j(ds'|s))$. Therefore, by the previous step, least and greatest Nash equilibrium $\underline{a}(v,s)$ and $\overline{a}(v,s)$ are increasing in v, for each $s \in S$ ■

In our next lemma, we show that for each extremal Nash equilibrium (for state s and continuation v), we can associate an equilibrium payoff that preserves monotonicity in v. To do this, we first compute the values of greatest (resp., least) best responses given a continuation values v and state s as follows:

$$\overline{\Pi}_i^*(v,s) := \Pi_i(v_i, s, \overline{a}_i(v,s), \overline{a}_{-i}(v,s))$$

and similarly

$$\underline{\Pi}_i^*(v,s) := \Pi_i(v_i, s, \underline{a}_i(v,s), \underline{a}_{-i}(v,s)).$$

We now have the following lemma:

Lemma 7.6 (Monotone values in G_v^s) *Under assumptions 2 and 3 we have: $\overline{\Pi}_i^*(v,s)$ and $\underline{\Pi}_i^*(v,s)$ are monotone in v.*

Proof of lemma 7.6: Function Π_i is increasing with $a_{-i}$and v_i. For $v_2 \geq v_1$by Lemma 7.5, we have $\underline{a}(v_2,s) \geq \underline{a}(v_1,s)$. Hence,

$$\begin{aligned}\underline{\Pi}_i^*(v^2,s) &= \max_{a_i \in \tilde{A}_i(s)} \Pi_i(v_i^2, s, a_i, \underline{a}_{-i}(v^2,s)) \geq \max_{a_i \in \tilde{A}_i(s)} \Pi_i(v_i^1, s, a_i, \underline{a}_{-i}(v^2,s)) \geq \\ &\geq \max_{a_i \in \tilde{A}_i(s)} \Pi_i(v_i^1, s, a_i, \underline{a}_{-i}(v^1,s)) = \Pi_i^*(v^1,s).\end{aligned}$$

A similar argument proves the monotonicity of $\overline{\Pi}_i^*(v,s)$. ■

To show that $\overline{T}(\cdot)(s) = \overline{\Pi}(\cdot,s)$ and $\underline{T}(\cdot)(s) = \underline{\Pi}(\cdot,s)$ are well-define transformations of $\mathcal{B}^n(S)$ we use the techniques introduced by Nowak and Raghavan (1992).

Lemma 7.7 (Measurable equilibria and values of G_v^s) *Under assumptions 2 and 3 we have:*

- $\overline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ *and* $\underline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$,
- *functions* $s \to \overline{a}(v,s)$ *and* $s \to \underline{a}(v,s)$ *are measurable for any* $v \in \mathcal{B}^n(S)$.

Proof of lemma 7.7: For $v \in \mathcal{B}^n(S)$ and $s \in S$, define the function $F_v : A \times S \to \mathbb{R}$ as follows:

$$F_v(a,s) = \sum_{i=1}^n \Pi_i(v_i,s,a) - \sum_{i=1}^n \max_{z_i \in \widetilde{A}_i(s)} \Pi_i(v_i,s,z_i,a_{-i}).$$

Observe $F_v(a,s) \leq 0$. Consider the problem:

$$\max_{a \in \times_i^n \tilde{A}_i(s)} F_v(a,s).$$

By assumption 2 and 3, the objective F_v is a Carathéodory function, and the (joint) feasible correspondence $\widetilde{A}(s) = \times_i \widetilde{A}_i(s)$ is weakly-measurable. By a standard measurable maximum theorem (e.g. theorem 18.19 in Aliprantis and Border (1999)), the correspondence $N_v : S \twoheadrightarrow \times_i A_i(s)$ defined as:

$$N_v(s) := \arg \max_{a \in \widetilde{A}_i(s)} F_v(a, s),$$

is measurable with nonempty compact values. Further, observe that $N_v(s)$, by definition, is a set of all Nash equilibria for the game G_v^s. Therefore, to finish the proof of our first assertion, for some player i, consider a problem $\max_{a \in N_v(s)} \Pi_i(v_i, s, a)$. Again, by the measurable maximum theorem, the value function $\overline{\Pi}_i^*(v, s)$ is measurable. A similar argument shows each $\underline{\Pi}_i^*(v, s)$ is measurable. Therefore, we have for value operators $\overline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$ and $\underline{T} : \mathcal{B}^n(S) \to \mathcal{B}^n(S)$.

To show the second assertion of the theorem, for some player i, again consider a problem of $\max_{a \in N_v(s)} a_i^j$ for some $j \in \{1, 2, \ldots, k_i\}$. Again, appealing to the measurable maximum theorem, the (maximizing) selection $\overline{a}(v, s)$ (respectively, $\underline{a}(v, s)$) is measurable with s. ■

Proof of theorem 4.1: Proof of 1. Clearly $\phi^1 \leq \phi^2$ and $v^1 \leq v^2$. Suppose $\phi^t \leq \phi^{t+1}$ and $v^t \leq v^{t+1}$. By the definition of the sequence $\{v^t\}$ and lemma 7.6 , we have $v^{t+1} \leq v^{t+2}$. Then, by Lemma 7.5, definition of $\{\phi^t\}$, and the induction hypotheses, we obtain $\phi^{t+1}(s) = \underline{a}(v^{t+1}, s) \leq \underline{a}(v^{t+2}, s) = \phi^{t+2}(s)$. Similarly, we obtain monotonicity of ψ^t and w^t.

Proof of 2: Clearly, the thesis is satisfied for $t = 1$. By induction, suppose that the thesis is satisfied for some t. Since $v^t \leq w^t$, by Lemma 7.6, we obtain

$$v^{t+1}(s) = \underline{\Pi}^*(v^t, s) \leq \underline{\Pi}^*(w^t, s) \leq \overline{\Pi}^*(w^t, s) = w^{t+1}(s).$$

Then, by Lemma 7.5, we obtain

$$\begin{aligned} \phi^{t+1}(s) &= \underline{a}(v^{t+1}, s) \\ &\leq \underline{a}(w^{t+1}, s) \quad \text{and hence} \\ &\leq \overline{a}(w^{t+1}, s) = \psi^{t+2}(s). \end{aligned}$$

Proof of 3-4: It is clear since for each $s \in S$, the sequences of values v^t, w^t and associated pure strategies ϕ^t and ψ^t are bounded. Further, by previous step, they are monotone.

Proof of 5: By definition of v^t and ϕ^t, we obtain

$$\begin{aligned} v_i^{t+1}(s) &= (1 - \beta_i) u_i(s, \phi^t(s)) + \beta_i \sum_{j=1}^{L} g_j(s, \phi^t(s)) \int_S v_i^t(s') \lambda_j(ds'|s) \\ &\geq (1 - \beta_i) u_i(s, a_i, \phi_{-i}^t(s)) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, \phi_{-i}^t(s)) \int_S v_i^t(s') \lambda_j(ds'|s), \end{aligned}$$

for arbitrary $a_i \in \tilde{A}_i(s)$. By the continuity of u_i and g and the Lebesgue Dominance Theorem, if we take a limit $t \to \infty$, we obtain

$$\begin{aligned} v_i^*(s) &= (1 - \beta_i) u_i(s, \phi^*(s)) + \beta_i \sum_{j=1}^{L} g_j(s, \phi^*(s)) \int_S v_i^*(s') \lambda_j(ds'|s) \\ &\geq (1 - \beta_i) u_i(s, a_i, \phi_{-i}^*(s)) + \beta_i \sum_{j=1}^{L} g_j(s, a_i, \phi_{-i}^*(s)) \int_S v_i^*(s') \lambda_j(ds'|s), \end{aligned}$$

which, by lemma 7.7, implies that ϕ^* is a pure stationary (measurable) Nash equilibrium, and v^* is its associated (measurable) equilibrium payoff. Analogously, we have ψ^* a pure strategy (measurable) Nash equilibrium, and w^* its associated (measurable) equilibrium payoff. ■

Proof of theorem 4.2: Step 1. We prove the desired inequality for equilibria payoffs. Since $0 \leq \omega^* \leq \overline{u}$, by Lemma 7.6 and definition of v^t and w^t, we obtain

$$v_1 \leq \omega^* \leq w_1.$$

By induction, let $v_t \leq \omega^* \leq w_t$. Again, from Lemma 7.6 we have:

$$\begin{aligned} v_{t+1} &= \underline{\Pi}^*(v_t, s) \leq \underline{\Pi}^*(\omega^*, s) \\ &\leq \omega^*(s) \leq \overline{\Pi}^*(\omega^*, s) \leq \overline{\Pi}^*(w_t, s) = w_{t+1}. \end{aligned}$$

Taking a limit with t we obtain desired inequality for equilibria payoffs.

Step 2: By previous step and Lemma 7.5, we obtain:

$$\begin{aligned} \phi^*(s) &= \underline{a}(v^*, s) \leq \underline{a}(\omega^*, s) \\ &\leq \gamma^*(s) \leq \overline{a}(\omega^*, s) \leq \overline{a}(w^*, s) = \psi^*. \end{aligned}$$

■

For fixed continuation value v let:

$$M_{i,j,l} := \sup_{s \in S, a \in A(s)} \frac{\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_l}}{-\sum_{\tilde{i}=1}^{n} \sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}},$$

$$M := \max \left\{ M_{i,j,l} : i = 1, \ldots, n, \text{ and } j = 1, \ldots, k_i, \text{ and } l = 1, \ldots, k \right\}.$$

By assumption 4, the constant M is a strictly positive real number.

Lemma 7.8 *Let assumptions 2, 3, 4 be satisfied and constraint functions $\tilde{a}_i \in CM^{k_i}$. Fix $v \in \mathcal{B}_n(S)$, and assume it is Lipschitz continuous. Consider an auxiliary game G_v^s. Then, there is a unique Nash equilibrium in this game $a^*(v, s)$ and belongs to $CM^{\sum_i k_i}$.*

Proof of lemma 7.8: Let $s > 0$ and let $v \in \mathcal{B}^n(S)$ be Lipschitz continuous function. To simplify we drop v from our notation. Let $x^1(s) = \tilde{a}(s)$ and $x_i^{t+1}(s) := \arg \max_{a_i \in \tilde{A}_i(s)} \Pi_i(s, a_i, x_{-i}^t(s))$ for $n \geq 1$. This is well defined by strict concavity of Π_i in a_i. Clearly, x^1 is nondecreasing and Lipshitz continuous with a constant less that M. By induction, assume that this thesis holds for $t \in \mathbb{N}$. Note that $(s, a_i) \to \Pi_i(s, a_i, x_{-i}^t(s))$ has increasing differences. Indeed if we take $s_1 \leq s_2$ and $y_1 \leq y_2$ then $x_{-i}^t(s_1) \leq x_{-i}^t(s_2)$ and

$$\begin{aligned} &\Pi_i(s_1, y_2, x_{-i}^t(s_1)) - \Pi_i(s_1, y_1, x_{-i}^t(s_1)), \\ &\leq \Pi_i(s_1, y_2, x_{-i}^t(s_2)) - \Pi_i(s_1, y_1, x_{-i}^t(s_2)), \\ &\leq \Pi_i(s_2, y_2, x_{-i}^t(s_2)) - \Pi_i(s_2, y_1, x_{-i}^t(s_2)). \end{aligned}$$

Therefore, since $\tilde{A}_i(\cdot)$ is ascending in the Veinott stron set order, by Theorem 6.1 in Topkis (1978) we obtain that $x_i^{t+1}(\cdot)$ is isotone. We show that $x_i^{t+1}(\cdot)$ is Lipshitz continuous with a constant M. To do this we check hypotheses of Theorem 2.4(ii) in Curtat (1996). Define $\varphi(s) = s_1 + \ldots + s_k$. Define $\mathbf{1}_i := (1, 1, \ldots, 1) \in \mathbb{R}^{k_i}$. We show that the function $(s, y) \to \Pi^*(s, y) := \Pi_i(s, M\varphi(s)\mathbf{1}_i - y, x_{-i}^t(s))$ has increasing differences. Note that $M\varphi(s) - \tilde{a}_i(s) \leq y \leq M\varphi(s)$. We show that the collection of the sets $Y(s) := [M\varphi(s) - \tilde{a}_i(s), M\varphi(s)]$ is ascending in the Veinott strong set order. Let $s_1 \geq s_2$ in product order Then,

$$M\varphi(s_1) - \tilde{a}_i^j(s_1) - \Big(M\varphi(s_2) - \tilde{a}_i^j(s_2)\Big) =$$
$$= M||s_1 - s_2||_1 - |\tilde{a}_i^j(s_1) - \tilde{a}_i^j(s_2)| \geq 0.$$

This, therefore, implies that lower bound of $Y(s)$ is increasing with s. Clearly upper bound of $Y(s)$ is increasing as well. Hence $Y(s)$ is ascending in the Veinott's strong set order.

Note that since for all $s_l \to x^t(s)$ is monotone and continuous, hence must be differentiable almost everywhere (Royden (1968)). By M Lipshitz property of x^t we conclude that each partial derivative is bounded by M. Hence we have for all $l = 1, \dots, k$, $i = 1, \dots, n$ and $j = 1, \dots, k_i$:

$$\frac{\partial \Pi_i^*}{\partial y_i^j} = -\frac{\partial \Pi_i}{\partial a_i^j}.$$

Next we have (for fixed s_{-k}):

$$\begin{aligned}
\frac{\partial^2 \Pi_i^*}{\partial y_i^j \partial s_k} &= -\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k} - M\sum_{\alpha=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a_i^j a_i^\alpha}\frac{\partial \varphi}{\partial s_k} - \sum_{\tilde{i}\neq i}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}\frac{\partial x^*_{\tilde{i},\tilde{j}}}{\partial s_k}. \\
&\geq -\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k} - M\sum_{\alpha=1}^{k_i} \frac{\partial^2 \Pi_i}{\partial a_i^j a_i^\alpha} - \sum_{\tilde{i}\neq i}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}} M \\
&= -\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k} - \sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}} M \\
&= \left(-\sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}\right)\left(M - \frac{\frac{\partial^2 \Pi_i}{\partial a_i^j \partial s_k}}{-\sum_{\tilde{i}=1}\sum_{\tilde{j}=1}^{k_{\tilde{i}}} \frac{\partial^2 \Pi_i}{\partial a_i^j \partial a_{\tilde{i}}^{\tilde{j}}}}\right) \geq 0
\end{aligned}$$

almost everywhere. Since $\frac{\partial \Pi_i^*}{\partial a_i^j}$ is continuous, by Theorem 6.2. in Topkis (1978) the solution of the optimization problem $y \to \Pi^*(v, s, y)$ (say $y^*(s, v)$) is isotone. From definition of $y^*(s, v)$ and x^{t+1} if $s_1 \leq s_2$ we have

$$0 \leq x_{i,j}^{t+1}(s_1) - x_{i,j}^{t+1}(s_2) \leq M(\varphi(s_1) - \varphi(s_2)) = M||s_1 - s_2||_1.$$

Analogously we prove appropriate inequality whenever $s_1 \geq s_2$. If s_1 and s_2 are incomparable then

$$x_{i,j}^{t+1}(s_1) - x_{i,j}^{t+1}(s_2) \leq x_{i,j}^{t+1}(s_1 \vee s_2) - x_{i,j}^{t+1}(s_1 \wedge s_2) \leq M||s_1 - s_2||_1$$

since $||s_1 - s_2||_1 = ||s_1 \vee s_2 - s_1 \wedge s_2||_1$. Similarly we prove that:

$$x_{i,j}^{t+1}(s_1) - x_{i,j}^{t+1}(s_2) \geq -M||s_1 - s_2||_1.$$

But this implies that x^{t+1} is also M-Lipshitz continuous, which implies that each x^t is M-Lipshitz continuous. Since Π_i has increasing differences in (a_i, a_{-i}) hence by Theorem 6.2 in Topkis (1978) we know that the operator $x \to \arg\max_{y_i \in \tilde{A}_i(s)} \Pi_i(s, y_i, x_{-i})$ is increasing. Therefore $x^t(s)$ must be decreasing in t. This implies that there exists $a^* = \lim_{n\to\infty} x^t$ which is isotone and M-Lipshitz continuous. Uniqueness of Nash Equilibria follows from assumption 4 and Gabay and Moulin (1980), hence $a^* = a^*(s, v)$ for $s > 0$.

Finally $\Pi(0, a) = 0$ for all $a \in A$ hence we can define $a^*(0) := \lim_{s\to 0^+} a^*(s)$ and obtain a unique Nash equilibrium $a^*(s, v)$ that is isotone and M-Lipschitz continuous in s. ■

Proof of theorem 4.3: To simplify notation, let $L = 1$ and hence $g(s,a) := g_1(s,a)$ and $\eta(f)(s) := \eta_1^f(s)$. Let $v \in \mathcal{B}^n(S)$ be Lipschitz continuous. Under assumptions 4, $a^*(s,v)$ is a well defined (for $s > 0$) as auxiliary game satisfies conditions of Gabay and Moulin (1980). Let $\pi_i(v_i,s,a) = (1-\beta_i)u_i(s,a) + \beta_i\eta_i(v_i)g(s,a)$, and observe that $\pi_i(v_i,s,\cdot)$ has also strict diagonal property, and obviously has cardinal complementarities. Here, note that

$$\mathcal{L}(\pi_i(v_i,s,\cdot)) = (1-\beta_i)\mathcal{L}(u_i(s,\cdot)) + \beta_i\eta_i(v_i)(s)\mathcal{L}(g(s,\cdot)) < 0.$$

Note further that applying Royden (1968) and continuity of the left side of expression below we have

$$\begin{aligned}\frac{\frac{\partial^2\pi_i(v,s,\cdot)}{\partial a_i^j \partial s_l}}{-\sum_{\tilde{i}=1}\sum_{j=1}^{k_{\tilde{i}}}\frac{\partial^2\pi_i(v,s,\cdot)}{\partial a_i^j \partial a_{\tilde{i}}^j}} &= \frac{(1-\beta_i)\frac{\partial u_i}{\partial a_i^j\partial s_l} + \beta_i\frac{\partial\eta(v)(s)}{\partial s_l}\frac{\partial g(s,a)}{\partial a_i^j} + \beta_i\eta(v)(s)\frac{\partial^2 g(s,a)}{\partial a_i^j\partial s_l}}{-(1-\beta_i)\mathcal{L}(u_i) - \beta_i\eta(v)(s)\mathcal{L}(g)} \\ &\le \frac{(1-\beta_i)U_{i,j,l}^2 + \beta_i\bar{\eta}G_{i,j}^1 + \beta_i\bar{u}G_{i,j,l}^2}{-(1-\beta_i)\mathcal{L}(u_i)} \le M_0.\end{aligned}$$

By Lemma 7.8, we know that $a^*(\cdot,v) \in CM_0^{\sum_i k_i}$. The following argument shows that $Tv(s)$ is Lipshitz continuous.

$$\begin{aligned}|T_iv(s_1) - T_iv(s_2)| &\le (1-\beta_i)|u_i(s_1,a^*(s_1,v)) - u_i(s_2,a^*(s_2,v))| \\ &+ \beta_i|\eta(v_i)(s_1) - \eta(v_i)(s_2)|g(s_1,a^*(s_1,v)) \\ &+ \beta_i\eta(v_i)(s_2)|g(s_1,a^*(s_1,v)) - g(s_2,a^*(s_2,v))| \\ &\le (U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u})||s_1 - s_2||_1 \\ &= M_1||s_1 - s_2||_1,\end{aligned}$$

where

$$U_1 := \sum_{l=1}^{h}\sup_{s\in S,a\in\tilde{A}(s)}\frac{\partial u_i}{\partial s_l}(s,a), \qquad U_2 := \sum_{i=1}^{m}\sum_{j=1}^{k_i}\sup_{s\in S,a\in\tilde{A}(s)}\frac{\partial u_i}{\partial a_i^j}(s,a),$$
$$G_1 := \sum_{l=1}^{h}\sup_{s\in S,a\in\tilde{A}(s)}\frac{\partial g}{\partial s_l}(s,a), \qquad G_2 := \sum_{i=1}^{m}\sum_{j=1}^{k_i}\sup_{s\in S,a\in\tilde{A}(s)}\frac{\partial g}{\partial a_i^j}(s,a),$$

and $M_1 := U_1 + M_0U_2 + \bar{\eta} + (G_1 + G_2M_0)\bar{u}$. Hence image of operator T is a subset of CM_1^n. Therefore the thesis is proven. ■

Proof of theorem 4.4: On CM^n define a function $T(v)(s) = \Pi^*(v,s)$. By a standard argument (e.g., Curtat (1996)) $T : CM^n \to CM^n$ is continuous and increasing on CM^n. By Tarski (1955) theorem, it therefore has a nonempty complete lattice of fixed points, say $FP(T)$. Further, for each fixed point $v^*(\cdot) \in FP(T)$, there is a corresponding unique stationary Markov Nash equilibrium $a^*(v^*,\cdot)$. ■

Lemma 7.9 *Let X be a lattice, Y be a poset. Assume (i) $F : X \times Y \to \mathbb{R}$ and $G : X \times Y \to \mathbb{R}$ have increasing differences, (ii) that $\forall\ y \in Y$, $G(\cdot,y)$ and $\gamma : Y \to \mathbb{R}$ are increasing functions. Then, function H defined by $H(x,y) = F(x,y) + \gamma(y)G(x,y)$ has increasing differences.*

Proof of lemma 7.9: Under the hypotheses of the lemma, it suffices to show that $\gamma(y)G(x,y)$ has increasing differences (as increasing differences is a cardinal property and closed under addition). Let $y_1 > y_2$, $x_1 > x_2$ and $(x_i, y_i) \in X \times Y$. By the hypothesis of increasing differences of G, and monotonicity of γ and $G(\cdot, y)$, we have the following inequality

$$\gamma(y_1)(G(x_1, y_1) - G(x_2, y_1)) \geq \gamma(y_2)(G(x_1, y_2) - G(x_2, y_2)).$$

Therefore,

$$\gamma(y_1)G(x_1, y_1) + \gamma(y_2)G(x_2, y_2) \geq \gamma(y_1)G(x_2, y_1) + \gamma(y_2)G(x_1, y_2).$$

■

Proof of theorem 4.5: Step 1. Let v_θ be a function $(s, \theta) \to v_\theta(s)$ that is increasing. By assumption 6 and lemma 7.9, the payoff function $\Pi_i(v_\theta, s, a, \theta)$ has increasing differences in (a_i, θ). Further, Π_i clearly also has increasing differences in (a_i, a_{-i}). As $\widetilde{A}_i()$ is ascending in Veinott's strong set order, by Theorem 6 in Milgrom and Roberts (1990), the greatest and the least Nash equilibrium in the supermodular game $G^s_{v,\theta}$ are increasing selections. Further, by the same argument as in Lemma 7.6, as $\widetilde{A}_i()$ is also ascending under set inclusion by assumption, we obtain monotonicity of corresponding equilibria payoff.

Step 2: Note, for each θ, the parameterized stochastic game satisfies conditions of Theorem 4.1. Further, noting the initial values of the sequence of $w^t_\theta(s)$ and $v^t_\theta(s)$ (constructed in Theorem 4.1) do not depend on θ and isotone in s, by the previous step, each iteration of both sequences of values is increasing with respect to (s, θ). Also, each of the iterations of $\phi^t_\theta(s)$ and $\psi^t_\theta(s)$ are also increasing in (s, θ). Therefore, as the pointwise partial order is closed, the limits of these sequences preserve this partial ordering, and the limits are increasing with respect to (s, θ). ■

For each equilibrium strategy f, define the operator

$$T^o_f(\eta)(A) = \int_S Q(A|s, f(s))\eta(ds). \tag{2}$$

where η^* is said to be invariant with respect to f if and only if it is a fixed point of T^o_f.

Proof of theorem 4.6: By theorem 4.5 both ϕ^* and ψ^* are increasing functions. By Assumption 3 $T^o_{\phi^*}(\eta)([s, \overline{S}]) = 1$ for $s \leq 0$ and for $s > 0$:

$$T^o_{\phi^*}(\eta)([s, \overline{S}]) = \sum_{j=1}^{L} g_j(s, \phi^*(s)) \int_S \lambda_j([s, \overline{S}]|s')\eta(ds').$$

Since by assumption, for each $s \in S$, the function under integral is increasing, the right-side is increase pointwise whenever η is stochastically increase. Moreover, as the family of probability measures on a compact state space S ordered by $\succeq$ (first order stochastic dominance) is chain complete (as it is a compact ordered topological space, e.g., Amann (1977), lemma 3.1 or corollary 3.2). Hence, $T^o_{\phi^*}$ satisfies conditions of Markowsky (1976) theorem (Theorem 9), and we conclude that the set of invariant distributions is a chain complete with greatest and least invariant distributions (see also Amann (1977), Theorem 3.3). By a similar construction, the same is true for the operator $T^o_{\psi^*}$.

To show the second assertion, we first prove that $T^o_{\phi^*}(\cdot)(A)$ is weakly continuous (i.e. if $\eta_t \to \eta$ weakly then $T^o_{\phi^*}(\eta_t) \to T^o_{\phi^*}(\eta)$ weakly). Let $\eta_t \to \eta$ weakly. By stochastic continuity of $\lambda_j(\cdot|s)$, we have $s' \to \lambda_j([s, \overline{S}]|s')$ continuous. Therefore,

$$\int_S \lambda_j([s, \overline{S}]|s')\eta_t(ds') \to \int_S \lambda_j([s, \overline{S}]|s')\eta(ds')$$

for all s. This, in turn, implies

$$T^o_{\phi^*}(\eta_t) \to T^o_{\phi^*}(\eta)$$

weakly. Let $\eta_t^{\phi^*}$ be a distribution of $s_t^{\phi^*}$ and $\eta_1^{\phi^*} = \delta_{\bar{S}}$. By the previous step, η_t is stochastically decreasing. It is, therefore, weakly convergent to some η^*. By continuity of T^o, we have $\eta^* = T^o_{\phi^*}(\eta^*)$. By definition of $\overline{\eta}(\phi^*)$, we immediately obtain $\overline{\eta}(\phi^*) \preceq \eta^*$. By the stochastic monotonicity of $T^o_{\phi^*}(\cdot)$, we can recursively obtain that $\delta_{\bar{S}} \succeq \eta_t^{\phi^*} \succeq \overline{\eta}(\phi^*)$, and hence $\eta^* \succeq \overline{\eta}(\phi^*)$. As a result, we conclude $\eta^* = \overline{\eta}(\phi^*)$. Similarly, we show convergence of the sequence of distributions $s_t^{\psi^*}$. ■

Proof of corollary 4.2: By theorem 4.6, there exists greatest fixed points for $T^{o,\theta_2}_{\phi^*}$ and $T^{o,\theta_1}_{\phi^*}$. Also, $T^{o,\theta}_{\phi^*}$ is weakly continuous. Further, $\theta \to T^{o,\theta}_{\phi^*}$ is an increasing map under first stochastic dominance on a chain complete poset of probability measures on the compact state set.

Consider a sequence of iterations from a $\delta_{\bar{S}}$ generated on $T^{o,\theta}_{\phi^*}$ (the operator defined in (2) but associated with $Q(\cdot|s,a,\theta)$). Observe, by Kantorvich-Tarski theorem (Dugundji and Granas (1982), theorem 4.2), we have

$$\sup_t T^{t,o,\theta_2}_{\phi^*} = \overline{\eta}_{\theta_2} \text{ and } \sup_t T^{t,o,\theta_2}_{\phi^*} = \overline{\eta}_{\theta_2}$$

As for any t, we also have $T^{t,o,\theta_2}_{\phi^*} \succeq T^{t,o,\theta_1}_{\phi^*}$. Therefore, by weak continuity (and the fact that $\succeq$ is a closed order), we obtain:

$$\overline{\eta}_{\theta_2} = \sup_t T^{o,\theta_2}_{\phi^*} \succeq \sup_t T^{t,o,\theta_1}_{\phi^*} = \overline{\eta}_{\theta_1}.$$

Similarly we proceed for ψ^*. ■

Remark 2 *Since norms $||\cdot||_1$ and $||\cdot||$ are equivalent on $\mathbb{R}^h$ and for all $s \in S$ $||s||_1 \leq h||s||$, hence we have that v^*, w^*, ϕ^* and ψ^* are Lipschitz continuous when we endow S with the maximum norm.*

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