Stationary Markovian equilibrium in Altruistic Stochastic OLG models with Limited Commitment[☆]

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Abstract

We introduce a new class of infinite horizon altruistic stochastic OLG models with capital and labor, but without commitment between the generations. Under mild regularity conditions, for economies with either bounded or unbounded state spaces, continuous monotone Markov perfect Nash equilibrium (henceforth MPNE) are shown to exist, and form an antichain. Further, for each such MPNE, we can also construct a corresponding stationary Markovian equilibrium invariant distribution. We then show for many versions of our economies found in applied work in macroeconomics, unique MPNE exist relative to the space of bounded measurable functions. We also relate all of our results to those obtained by promised utility/continuation methods based upon the work of Abreu et al. (1990). As our results are constructive, we can provide characterizations of numerical methods for approximating MPNE, and we construct error bounds. Finally, we provide a series of examples to show the potential applications and limitations of our results.

[☆]We thank Rabah Amir, Manjira Datta, Len Mirman, Adrian Peralta-Alva, and Ed Prescott for helpful conversations as well as an anonymous referee for important points. We also thank participants of our sessions at XIXth European Workshop on General Equilibrium Theory, Kraków, as well as at the 4th International Conference on Game Theory and Management, St. Petersburg for helpful comments. Reffett thanks the Dean's Summer Grant Program at ASU for generous financial support of this research. All the usual caveats apply.

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Keywords: stochastic games, Markov perfect Nash equilibrium, constructive methods, commitment
JEL codes: C62, C73, D91, 040

1. Introduction and related literature

Over the last two decades, there has been a renewed interest in studying dynamic general equilibrium models without commitment. The seminal examples of such dynamic and time consistency problems that arise in economics are found in the work of Kydland and Prescott (1977, 1980) and Levhari and Mirman (1980). More recent work along these lines has included models of sustainable plans, altruistic growth, Ricardian equivalence, endogenous borrowing constraints, sovereign debt, monetary policy games, savings with hyperbolic discounting, and Ramsey taxation. A central issue that has emerged in this literature is how one can develop appropriate versions of these models such that one can (i) obtain sufficiently rich characterizations of the structure of the set of subgame perfect equilibria that arise, while (ii) developing rigorous numerical methods that can be used to compute this set.

Many methodological proposals for solving these models with commitment frictions have been made. For example, beginning with the work of Kydland and Prescott (1977, 1980) on time consistent optimal Ramsey taxation, and continuing in many recent papers (e.g., Atkeson (1991), Chari and Kehoe (1993), Sleet (1998), and Phelan and Stacchetti (2001), among others), the so-called "promised utility" approach for constructing subgame perfect equilibrium has been proposed. In this method, one constructs a set of sustainable values for each player in a subgame perfect equilibrium by applying strategic dynamic programming arguments.[2] In related work, others have appealed to modifications of traditional dynamic programming approaches, where particular subgame perfect equilibrium are studied (namely, Markov perfect Nash equilibrium (MPNE)). The existence of such (pure strategy) equilibrium has typically been studied using "direct approaches" based upon verifying the existence of a fixed point of best response maps, or using a more

[2] It bears mentioning that in Kydland and Prescott's original work, time-consistent policies were the focus. In this later work in promised utility methods, Markov perfection was not necessarily the focus. For an interesting survey of strategic dynamic programming methods, see the work of Pearce and Stacchetti (1997) and Sleet and Yeltekin (2003).

”indirect approach”, that introduces (dynamic) incentive constraints and dual state variables (each with a recursive structure), that tries to compute incentive constrained equilibrium as the outcome of a parameterized ”recursive saddlepoint” problem. Typical work taking the former approach (for constructing MPNE) has been implemented in a number of papers (e.g., see, Alj and Haurie (1983), Amir (1989, 1996b, 2002, 2005), Curtat (1996), and Nowak (2006)), while in the more indirect approach for cases where a ”worst” equilibrium is easily defined and formalize as a set of dynamic incentive constraints have been also proposed (e.g., the methods of Marcet and Marimon (2009), Rustichini (1998), and Messner et al. (2011)). Finally, new Euler equations methods have been proposed in the work of Harris and Laibson (2001) and Klein et al. (2008), where a first order theory for MPNE via a ”generalized” Euler equation approach (GEE) is developed.

Although these approaches are each promising, they are also known to suffer from some well-known technical limitations. For example, when developing a promised utility approach, it is often very difficult to characterize (rigorously) the set of equilibrium pure strategies that sustain the set of subgame perfect equilibrium values. Further, these sustainable values are argued to be some appropriate selection from the APS value equilibrium correspondence. Finally, these methods often require discounting (which will not be the case in the models we study). When applying more direct traditional dynamic programming methods, one faces a serious problem of finding a suitable function space to solve the fixed point problem associated with existence of MPNE. Further, for all of these dynamic programming style approaches (either direct or indirect), the methods typically provide very weak characterizations of the *set* of subgame or Markov perfect Nash equilibrium, or conditions under which uniqueness of stationary Markovian equilibrium can also arise. This is an important limitation for many applications. For example, one needs such a sharp characterization of equilibrium (and equilibrium stability in parameter) when (i) econometricians seek to estimate such dynamic models, or (ii) economists seek to calibrate the equilibrium of such models to economic data.

Also, when using the dynamic programming methods with recursive incentive constraints, where dual variables are introduced as state variables (e.g., as in Marcet and Marimon (2009), Rustichini (1998) and Messner et al. (2011)), a serious concern is the nature of the punishment schemes that are used to sustain subgame perfect equilibrium (as by their nature, they are imposed in an *ad hoc* manner). These sorts of procedures amount to equilibrium selection devices at one level. But, perhaps more troubling for some of these approaches (e.g., Marcet and Marimon recursive saddlepoint methods), the important counterexamples exist, which appear when dual programs are not strictly concave. Finally, when applying GEE methods, the question of relating the first order theory (and the assumed smoothness of equilibrium solutions) to the equilibrium value function that solve agents dynamic programs in the game have yet to be rigorously developed.[3] So a deep understanding of how these collections of methods work has yet to be firmly established.

The central purpose of this paper is to address many of these concerns in the context of a stochastic altruistic growth models without commitment and with elastic labor supply. The class of models we study are related to those first introduced in Amir (1996b) and Nowak (2006), (i.e. models of stochastic growth without commitment), but they possess additional many interesting complications that are associated with multidimensional action spaces (e.g., joint decisions of each generation each period on both capital and labor). The deterministic incarnation of our models have a long history, dating back to the work of Phelps and Pollak (1968), and have been studied extensively in the recent literature (e.g., Alj and Haurie (1983), Amir (1996b), Nowak (2006), as well as the references therewithin). The economy we study consists of a sequence of identical generations, each living one period, each deriving utility from its own consumption and leisure as well as the consumption and leisure of its successor generation. In our models, current generations are faced with the problem that future generations cannot commit to actions, so each current generation seeks to implement dynamic plans as subgame (or Markov) perfect equilibrium. Given the lack of commitment, this situation is non-trivial, as agents in this environment face a time-consistency problem each period, as the current generation has an incentive to deviate from given sequence of bequests, consume a disproportionate amount of current bequests, leaving little (or nothing) for subsequent generations.

A key methodological development found in our approach to this problem is to pose the problem as a stochastic game, where we introduce paternalistic

[3] For example, Klein et al. (2008) apply the implicit function theorem at the steady-state on the agents Euler equation to construct a local GEE. Unfortunately, it is not proven that on the open set near this steady state, the Euler equation is sufficient (as the equilibrium value function need not be concave). Similar issues arise in the first order theory of Harris and Laibson (2001).

altruism over two objects (namely, the next generations consumption-leisure pair). This extension is nontrivial, yet important for applications, as a great deal of work in macroeconomics and dynamic public finance takes place in the context of frictions between generations (e.g., models of government finance and taxation, environmental control, etc.), where agents have preferences (resp., technologies) defined over consumption and leisure (respectively, capital and labor) over their lifecycle. Although this feature does complicate matters a great deal, we are still able to obtain existence results under very general conditions. Further, we are able to give conditions under which the uniqueness results obtained in Balbus et al. (2009) can be extended to this class of models under reasonable sufficient conditions (namely, conditions involving elasticities of preferences over consumption and leisure). We also discuss the possibility of extending our results to models where agents are longer-lived.

We prove many results about the class of models studied in this paper. We first address the question of existence of MPNE. Although existence of MPNE has been studied in versions of the model with inelastic labor supply in both the deterministic (e.g., Leininger (1986), Kohlberg (1976)) and stochastic setting (Amir (1996b), Nowak (2006), Balbus et al. (2009)), these proofs do not apply the case of elastic labor supply and more general paternalistic altruism. Moreover in a related multi-period, stochastic OLG models important counterexamples to existence of Markovian (recursive) equilibrium are known (Kubler and Polemarchakis, 2004; Citanna and Siconolfi, 2008). Therefore, the question of existence itself is very important. Next, we ask the question of when sufficient conditions exist under which globally stable constructive iterative procedures can be employed for computing *unique* MPNE. Such conditions are obviously of interest for the numerical studies. We give reasonable conditions for uniqueness of MPNE relative to a very broad class of potential equilibria (namely, we prove uniqueness relative to the space of all bounded, Borel measurable strategies, and existence in the space of continuous functions). Under conditions on primitives that guarantee uniqueness, our global stability result for iterative methods applies relative to any initial choice of Borel measurable function that is pointwise feasible in the game.

We next turn to the question of existence of Stationary Markov equilibrium (SME) in the game (i.e., a characterization of the set of equilibrium invariant distributions associated with the set of MPNE), as well as results on the computation of MPNE and SME. As the question has received very little attention in the literature, for our games, we give conditions for existence of SME, as well as provide conditions for uniqueness of corresponding nontrivial invariant distributions. These existence and stochastic stability results hold for both the case of bounded or unbounded state spaces. We next turn to the question of theoretical characterizations of numerical implementations. As all our methods are constructive, we are able to provide two approximation results that allow us to construct error bounds of MPNE set, and provide a rigorous numerical approach to approximating the unique MPNE (including methods that achieve uniform error bounds). These error bounds in some cases are sharp, and relative to competing methodological approaches, our numerical methods are quite simple to implement. Given our results on SME, we are able to provide a set of sufficient conditions under which globally stable approximation schemes can be provided for our class of stochastic games both relative to MPNE and SME.

The paper concludes with a discussion of extensions of our results to models where agents live (or interact) over multiple periods, as well as to some problems in areas such as environmental economics and public finance. We first discuss possible extensions of the model, and discuss when our methods work, and what limitations of our results arise in multiperiod frameworks. In particular, we discuss the possibility of extending our methods to dynamic models with (i) multiperiod altruism, or (ii) multiperiod-lived agents. We provide examples of economies where our existence results for MPNE hold, as well as an example of a multiperiod-lived agent OLG model where they fail (even for existence). Further, and perhaps most importantly, we also show that in cases where our existence argument still apply, we show why our computational approaches can fail. In response to this situation, we then discuss in detail the possibility of developing an APS method in function spaces to construct sustainable (by MPNE) values, and provide a new method for constructing dynamic equilibrium in our models where agents either have multiperiod altruism or are multiperiod-lived. This new APS methods remains constructive, but works in different partial orders (namely set inclusion, as opposed to ”pointwise” partial orders).

The results in this paper are important for numerous reasons. First, we extend the set of existence results for stochastic OLG models without commitment in an important way to models with multidimensional altruism, as well as models with other dynamic commitment problems, multiperiod altruism, and multiperiod-lived agents. We also provide a new set of simple tools that can form the basis of accurate quantitative studies of stochastic OLG economies with limited commitment.

Second, our methods work in very general settings, and, therefore have the potential to shed new light on other dynamic economies with dynamic inconsistencies (e.g., Phelan and Stacchetti (2001); Atkeson (1991)), models with hyperbolic discounting (like Peleg and Yaari (1973); Krusell and Smith (2003)), games of multigenerational altruism with capital accumulation, fishwars, or other dynamic resource extraction problems (Levhari and Mirman, 1980), and more general stochastic discounted supermodular games (like Curtat (1996); Amir (2002)), among others.

Third, from a technical perspective, we build upon the monotone approach first proposed in Balbus et al. (2009) for stochastic growth models with related commitment frictions. A critical aspect of this approach is that unlike work in the existing literature on monotone methods built upon increasing operators, we build our theory of existence and computation upon the computation of fixed points of *decreasing* operators. Further, when comparing our methods to those used to study recursive equilibrium in one sector nonoptimal stochastic growth models in case of elastic labor supply with perfect commitment (e.g., Coleman (1997) and Datta et al. (2002)), as we are forced to work with decreasing operators, even the question of existence of fixed points is a much more complicated (as opposed to the case of increasing operators where existence can be established via various versions of Tarski's theorem). Also, relative to uniqueness conditions, our methods in this paper are also related to those in Balbus et al. (2009), where the fixed point theorems used to show uniqueness are based on geometrical properties of monotone mappings defined in abstract cones found in the work of Guo and Lakshmikantham (1988) or Guo et al. (2004). But as will be clear from the paper, the existence of multidimensional altruism (e.g., altruism over both successor generation consumption and leisure) greatly complicates the characterization of sufficient conditions for global stability.

The rest of the paper is organized as follows: in section 2 we present the formal model and state our assumptions, in section 3 we state our main results, and some examples of applications of our results. Section 4 states our result on existence/uniqueness as well as approximation of invariant distribution and hence Stationary Markovian Equilibria for our economy. In section 5 we discuss how our results can be extended to stochastic OLG economies with multiperiod structure and limited commitment. In the next section, we provide some applications of the results to other related economies including those with (i) human capital and public education, (ii) environmental quality and (iii) models of endogenous fertility. That last section 7 gives the proofs of all the theorems in the paper.

2. The class of Economies

Our benchline model is a stochastic overlapping generations production economy with capital and labor, but without commitment between successor generations. Time is discrete and indexed by $t = \{0, 1, 2, \ldots\}$. The economy is populated by a sequence of identical short-lived agents (or "generations"), each living one period, each caring about her own consumption of output goods and leisure, as well as that of its successor generation. Given our indexation of time, we shall study a sequential infinite horizon stochastic game with a countable number of players whose "names" are indexed by time t. For simplicity, the size of each generation is assumed to be equal and normalized to unity, and there is no population growth. Apart from elastically supplying labor services $1 - l$ (where l denotes leisure), any given generation divides its (inherited) finished output goods s between current consumption c and investment in a stochastic production technology $x = s - c$, with the proceeds of this investment being left to the next generation as a bequest. As is often typical for OLG models in macroeconomic applications, each new generation born has time separable utility, and receives lifetime utility from both their own consumption and leisure today (c, l) as well as the expected utility of its immediate successor consumption and leisure (c', l'). As is standard in the class of models we study, there is a stochastic production technology summarized by stochastic transition Q, but in our case, this technology will map current investment savings, labor supply, and output $(s - c, 1 - l, s)$ into next period output[4] denoted by s'.

Let K be a set of possible capital stocks, and $L := [0, 1]$ be a set of possible levels of labor (where we have normalized each generations endowment of time to the unit interval). We shall consider two cases for the capital stock, namely the unbounded state space case (i.e., $K := \mathbb{R}_+$), and the bounded case (i.e., $K := [0, S]$, where $S \in \mathbb{R}_{++}$). For any stationary, measurable con-

[4] Observe that such a formulation of a stochastic technology allows for many specific models as special cases (see also Amir (1997)). For example, it may include a within period consumption good production technology f depending on $(1 - l, s)$. One (insignificant) limitation is that, in our specification production takes time after consumption. This is without reducing generality of our results, however, as tools prosed in this paper allow for different (within period) timing schedules.

tinuation policy of the next generation for consumption and leisure denoted by $h := (h_1, h_2)$ (with $h : K \to K \times L$), the objective function for the current generation is well defined, and given by:

$$U(c, l; h, s) := u(c, l) + \int_K v(h_1(y), h_2(y))Q(dy|s - c, 1 - l, s),$$

where $u(c, l)$ is the current generations utility from its own consumption and leisure, and $v(c', l')$ is the altruistic utility the current generation receives from the consumption and leisure of the successor generation next period.

Before stating our assumptions on the primitives of this environment, we first introduce some basic notation that shall be used throughout the paper. For each state $s \in K$, let the correspondence $A(s) := \{(c, l) \in K \times L, c \leq s\}$ denote the set of feasible actions for any current generation in s, and $\text{Int}(A(s))$ denote interior of this set. Also, let δ_0 denote a probability measure on K concentrated at point zero.

We now state a series of assumptions on preferences and stochastic production that we shall use for our existence results:

Assumption 1 (Preferences 1). *The functions u and v satisfy the following:*

- $u : K \times L \to \mathbb{R}_+$ *is twice continuously differentiable, strictly increasing in both arguments, supermodular and strictly concave function,*
- $v : K \times L \to \mathbb{R}_+$ *is increasing, measurable and* $\int_K v(s', 1)\lambda_k(ds'|s) < \infty$ *for all* $k = 1, \ldots, m$ *where the measures* λ_k *will be specified later*[5]*,*
- $v(0, l) = 0$.

As a special case of these preferences, we shall often consider the case of additive separability between current consumption and leisure. This is actually the typical assumptions used in applications in macroeconomics.

Assumption 2 (Preferences 2). *The functions u and v satisfy:*

[5]The exact details of the assumptions on the measures λ_k are spelled out in Assumptions 3, 4, and 5 below.

- *$u(c,l) = u_1(c) + u_2(l)$, where u_1 and u_2 are twice continuously differentiable, strictly increasing, strictly concave functions and*

- *$u_1(0) = u_2(0) = 0$,*

- *for $i = 1, 2$ the function u_i satisfies $u_i'(0^+) = \infty$,*

- *$v : K \times L \to \mathbb{R}_+$ is increasing, measurable and $\int\limits_K v(s', 1)\lambda_k(ds'|s) < \infty$ for all $k = 1, \ldots, m$, (where measures λ_k would be specified later),*

- *$v(0, l) = 0$.*

The model of stochastic production that we adopt is a special case of the one studied in a series of papers by Amir (1996a,b, 1997), Nowak and Szajowski (2003), Balbus and Nowak (2004, 2008), Nowak (2006), and related to the model of production in Magill and Quinzii (2009). For our purposes, it proves to be convenient to assume a version of the mixing formulation for the stochastic transitions that has been the focus of extensive study in the literature (e.g., see discussion in Nowak (2006), and references therewithin). For our present work, we extend this stochastic production specification to the case of elastic labor supply. It turns out for our methods, it suffices to assume that stochastic production is a mixture of two probability measures that admit densities:

Assumption 3 (Transition: general case). *In the model without absorbing state, let transition*

- *Q be given by:*

  $$Q(\cdot|s - c, 1 - l, s) := g(s - c, 1 - l)\lambda_1(\cdot|s) + (1 - g(s - c, 1 - l))\lambda_2(\cdot|s),$$

  where the function $g : K \times L \to [0, 1]$ is twice continuously differentiable, increasing in both arguments, supermodular and concave with $0 \leq g(\cdot) \leq 1$,

- *we have $(\forall c, l \in A(s))\, g(c, 0) = g(0, l) = 0$,*

- *there exists a measure μ such that each measure $\lambda_k(\cdot|s)$ has a density $\rho_k(\cdot, s)$ with respect to a common measure μ, i.e. can be described as $\lambda_k(A|s) = \int\limits_A \rho_k(s', s)\mu(ds')$.*

For some of the results in the paper, we shall consider two modifications of this assumption for the stochastic transition Q. The first modification allows for an absorbing state, while the second imposes additional additive separability in the structure of Q. These two different cases are given in the following two assumptions.

Assumption 4 (Transition: absorbing state). *Let transition*

- Q *be given by:*

$$Q(\cdot|s-c,1-l,s) := \sum_{i=1}^{m} g_i(s-c,1-l)\lambda_i(\cdot|s) + g_0(s-c,1-l)\delta_0(\cdot),$$

  where for $i=1,\ldots,m$ *functions* $g_i : K \times L \to [0,1]$ *are twice continuously differentiable, increasing in both arguments, supermodular, concave with* $0 \leq g_i(\cdot) \leq 1$ *and* $\sum_{i=0}^{m} g_i(\cdot) = 1$,

- *there exists a measure* μ *such that each measure* $\lambda_k(\cdot|s)$ *has a density* $\rho_k(\cdot,s)$ *with respect to a common measure* μ, *i.e. can be described as* $\lambda(A|s) = \int_A \rho_k(s',s)\mu(ds')$.

Assumption 5 (Transition: separated variables). *Let transition*

- Q *be given by:*

$$Q(\cdot|s-c,1-l,s) := g(s-c,1-l)\lambda(\cdot|s) + (1-g(s-c,1-l))\delta_0(\cdot),$$

  where the function $g : K \times L \to [0,1]$ *is of the form* $g(a,b) = g_1(a) + g_2(b)$ *and each* g_i *is twice continuously differentiable, strictly concave and strictly increasing on* K *with* $0 \leq g(\cdot,\cdot) \leq 1$ *and* $g_1(0) = g_2(0) = 0$,

- *there exists a measure* μ *such that* $\lambda(\cdot|s)$ *has a density* $\rho(\cdot,s)$ *with respect to measure* μ, *i.e. can be described as* $\lambda(A|s) = \int_A \rho(s',s)\mu(ds')$,

- *moreover the collection of the measures* $\lambda(\cdot|s)$ *is stochastically decreasing with* s *on* K.

A few remarks on the assumptions are warranted. As our assumptions on preferences are completely standard, we shall focus our attention on the assumptions on transition structure for stochastic production, as well as the role of the state space in that specification. In related work by Nowak (2006) or Balbus et al. (2009), the authors assume that the state space K is a compact interval in $\mathbb{R}_+$. This differs from Amir (1996b), where the state space is taken to be unbounded $K = \mathbb{R}_+$. In the latter paper, to show the existence of MPNE, it is critical that the space space is unbounded; this is not the case of Nowak (2006) or our present methods.

Additionally, following Nowak (2006), our transition Q is a convex combination of a finite number of measures λ_i (in assumption 4 and 5 also δ_0), each depending jointly on the state s, as well as the decision variables $(s - c, l)$. In this specification, the functions g_i can be viewed as the "weights" placed on probability measures that govern the stochastic process governing the structure of production. In what follows, we shall analyze cases of stochastic production both with and without the presence of an absorbing state. More specifically, the former case of an absorbing state is obtained by taking one of the measures (namely δ_0) to be a delta Dirac measure concentrated at point zero. The examples of transitions satisfying these assumptions (but without elastic labor supply) can be found in Nowak (2006). Also, we should mention that the supermodularity assumptions on the primitives of preferences u and production g are critical for showing monotonicity of a best response operator in a model with an absorbing state.

We next discuss the case of stochastic production without an absorbing state. First note that in our approach, we can work with either bounded and unbounded state spaces. One reason we must allow for both cases stems from an observation first made by Balbus et al. (2009) that concerns the possibility of the degeneracy of stationary Markov equilibrium (SME) that can arise in models with bounded state spaces and stochastic production with absorbing states. In particular, if one assumes the following: (i) bounded state space, (ii) existence of an absorbing state 0, strict monotonicity of g, and (iii) interiority of a MPNE, this implies a positive probability of reaching an absorbing state 0 each period. This implies that the SME has a trivial invariant distribution from all initial states. By allowing for unbounded state spaces, we can provide situations where the stochastic transitions for production with and without absorbing states allow us to obtain conditions where we can avoid this trivial outcome for SME. To do this, we require only weak monotonicity of functions g_i, and hence we can reduce probability of reaching state zero to infinitesimal. Finally, it bears mentioning that as we do not specify measures λ_i, we can obtain many strictly positive absorbing states. We should also remark that in the existing literature, very little, if any, attention has been focused on the structure of stationary Markov equilibrium in the class of games studied. One exception, being Balbus et al. (2009) for the case of inelastic labor, and a more restrictive form of intergenerational altruism.

Remark 1. Modeling production uncertainty through the probability distribution it induces on the outcome space is perhaps nontypical in macroeconomic applications. Stokey et al. (1989, chp. 8-9) describe uncertainty through a function of the random variable, i.e. as a map from a state space to the real line. In the former approach probabilities of outcomes are endogenous while in the latter probabilities of states (typically not outcomes) are exogenous. Technically it can be shown that both descriptions are equivalent, i.e. one can be represented by the other. The assumptions needed to guarantee existence of equilibria (in economies and most importantly in games), however, are typically weaker for the probability distribution approach (see Amir (1997) for examples and counterexamples). The reason is that probability distribution approach uses all the convexity-type properties of the transition probability. The second reason for using such description follows from Magill and Quinzii (2009), who argue that it is indeed more natural representation of the real economy. Hence although we can redefine our transition probability using functions of a random variables (see for such reformulation in section 7.4), it is not clear that results obtained in this paper for a dynamic economy would still hold.

To understand our assumptions in the context of the existing literature more precisely, in related work on stochastic bequest economies (with inelastic labor supply), the paper of Amir (1996b) uses a different approach to characterizing the stochastic transition Q. Apart on the assumptions of the state space K that we have already discussed (i.e., K must be unbounded), the main differences between our case (following Nowak) and Amir concern assumptions on the particulars of stochastic production. First, Amir (1996b) assumes that transition Q parameterized by current decisions, is (weakly) continuous, stochastically increasing and stochastically concave, while Nowak (2006) takes Q to depend on both current decisions and current state, and lets Q be given by a convex combination of a finite number of measures, where weights are given by the production process g_i. Therefore, on the one hand, Nowak (2006) does not require stochastic monotonicity and stochastic concavity of Q, while on the other hand, Amir (1996b) does not require the particular mixing structure for Q.

Many of the critical results obtained in this paper follow from the monotonicity of a best response operator, where sufficient conditions are given for mixing structure of Q as studied in Nowak. It is not clear how such results can be easily generalized to the case of stochastic transition structure on Q of Amir's form. One way to understand this tension is the following: if we think of a transition given by assumption 3 when a measures λ_1 is stochastically dominating λ_2, we can generate Amir's transitions (e.g., see his example 2 and his related comments), but unfortunately this set of assumptions is not sufficient to show monotonicity of the best response operator that we study. This will, therefore, imply this condition is not sufficient for uniqueness of MPNE using methods developed in this paper.

We now can define a (pure strategy) MPNE. By D, we shall denote a set of all bounded (by 0 and S), measurable pure strategies, i.e.:

$$D := \{h : K \to K \times L : \forall_{s \in S} h(s) \in A(s), h \text{ is bounded and measurable}\}$$

endowed with the topology of uniform convergence on compact subsets of S (i.e. equivalent to sup-norm topology for compact S) and standard pointwise (product) order $\leq$ by:

$$(\forall \xi, \eta \in D) \quad \xi \leq \eta \quad \text{iff } (\forall s \in K)\, \xi_1(s) \leq \eta_1(s) \quad \text{and} \quad \xi_2(s) \leq \eta_2(s).$$

For $h \in D$, given continuity of the primitive data of the model (i.e., utility and stochastic production), be a simple application of Berge's maximum theorem, we can define the best response map BR as follows:

$$BR(h)(s) := \arg \max_{(c,l) \in A(s)} U(c, l; h, s).$$

Then, a *Markov perfect Nash equilibrium* (MPNE) in D is any function $h^* \in D$ such that $h^* \in BR(h^*)$.

3. Existence and approximation of MPNE

We first address the issue of sufficient conditions for existence. That is, as it is well-known that in stochastic OLG models with limited (Amir, 1996b; Nowak, 2006) or full commitment (Kubler and Polemarchakis, 2004; Citanna and Siconolfi, 2008, 2010), existence of Markovian (or recursive) equilibria is not generally guaranteed, it is important we first provide conditions under which such MPNE exist. As we consider different assumptions on stochastic production in Assumptions 3, 4 and 5, we divide the following discussion into three subsections.

3.1. Existence in Models with an Absorbing State

We begin our study of MPNE by considering the case of stochastic production with an absorbing state. In this setting, we first prove the existence of MPNE in the set of bounded, measurable strategies D. In particular, under assumptions 1 and 4, we can write the objective for a typical generation as:

$$U(c,l;h,s) := u(c,l) + \sum_{k=1}^{m} \int_{K} v(h_1(y), h_2(y))\lambda_k(dy|s)g_k(s-c, 1-l).$$

In Theorem 1, we state our first major existence result.

Theorem 1 (Existence of MPNE). *Under assumptions 1 and 4, there exists a MPNE. Moreover, the set of MPNE in D forms an anti-chain (i.e. has no ordered elements).*

First, notice that the existence result in theorem 1 (and later, in 4) is obtained under very general conditions on technology and preferences. These results extend the existence results previously obtained in Amir (1996b) and Nowak (2006) to the case of altruistic stochastic growth with much more general paternalistic altruism (namely, altruism defined over multidimensional continuation strategies used by the successor generation). We shall stress that although our MPNE existence result can be viewed as extending the application of the tools of Amir (1996b) (for unbounded state space and Lipschitz continuous MPNE) and Nowak (2006) (for bounded state space and Borel measurable MPNE) to models with multidimensional strategic altruism, the key aspect of our existence theorems is that we can also provide an characterization of the MPNE set (namely, that it is a nonempty antichain). To obtain this result, it is critical to study the fixed point of decreasing operators, and is novel to our approach exploiting our assumptions on stochastic transition Q.

The intuition behind the proof of theorem 1 is useful to discuss. Under assumptions 1 and 4, one can write down a standard first order condition for a (two dimensional, single-valued) best response of the current generation to the continuation strategy of the successor generation, and given the concavity of the problem under the transition Q, this Euler equation is both necessary and sufficient. Further, the mixing assumption 4 guarantees that the best response operator is also decreasing. Therefore, by a standard result, its fixed point set (if nonempty) must be anti-chained. To prove the fixed point set is nonempty, we apply the Schauder-Tikhonov fixed point theorem to a continuous best response operator mapping a (weak-*) compact set of mixed strategies to itself. Under assumptions 1 and 4, the payoff function is strictly concave (see Amir (1996b) or Nowak (2006) for similar results) for any continuation mixed strategy of the next generation; hence, the existence of pure-strategy MSNE is guaranteed.

Finally, observe that unlike work for recursive competitive equilibrium for competitive economies (e.g., Kubler and Polemarchakis (2004) or other papers with examples for non-existence of recursive competitive equilibria such as discussed in Citanna and Siconolfi (2008)), each generation's within period equilibrium (or best response) allocation for a given continuation strategy of the following generation is unique. Hence, due to stochastic concavity of our noise specification (among others), we avoid the difficult equilibrium selection problem observed by Hellwig (1983), which is used to construct counterexamples for existence of recursive equilibrium in the OLG models studied in Kubler and Polemarchakis (2004). It does bear mentioning, though, that by adding specific endogenous noise on the future realization of states, our result has a flavor of the argument of Citanna and Siconolfi (2008, 2010), who are able to restore generic existence of recursive equilibria by perturbing utility of the old generation, while allowing for sufficient heterogeneity of individuals within each generation. A key difference between our and this related work is that we consider these question in the context case of stochastic growth with limited commitment, and for these models, many important decentralization issues associated with recursive competitive equilibrium are not known (and, hence, not the focal point of our analysis).[6]

[6]In current work, we are developing methods to decentralize the MPNE in this environment as a recursive competitive equilibrium using a duality methods for stochastically concave constrained dynamic optimization models. So it does appear possible in models with limited commitment such as the present to first construct a MPNE (or sequential)

We now mention a few corollaries of theorem 1. In particular, we consider the further characterization of the continuity and monotonicity properties of MPNE policies. We begin in corollary 1 with an additional result on the continuity properties of MPNE.

Corollary 1 (Continuous MPNE). *Let assumptions 1 and 4 be satisfied. Assume additionally that*

i) there exists a μ-measurable function $\bar{\rho}$ such that $\rho_j(s', s) \leq \bar{\rho}(s')$ for each $s', s \in K$ and $j = 1, ..., m$,

ii) for each function $f : K \to K$ such that $\int\limits_K f(s')\bar{\rho}(s')\mu(ds') < \infty$ the integral $\int\limits_K f(s')\rho_j(s', s)\mu(ds')$ is continuous as a function of s,

iii) $\int\limits_K v(s', 1)\bar{\rho}(s')\mu(ds') < \infty$.

Then, there exists a MPNE (c^*, l^*) where $c^*(\cdot)$ and $l^*(\cdot)$ are continuous functions.

Notice, unlike the results in the case of inelastic labor supply (e.g., Amir (1996b)), our result only establishes the existence of *continuous* MPNE. That is, for dynastic stochastic growth models without commitment with more general forms of intergenerational altruism (e.g., over both consumption and leisure), one is not able to obtain MPNE that are Lipschitzian (see discussion below). In particular, with elastic labor supply, investment decisions are continuous, but *not* increasing everywhere. This result is also in contrast to the case of stochastic growth models with elastic labor supply and perfect commitment (where investment can be show to be increasing in the current period capital stock/output).[7]

equilibrium, and then ”price” that equilibrium under an suitable price system adapted to the stochastic production setting of this model. The question of relating MPNE in our environment to such recursive equilibrium in OLG models with stochastic production will be explored in subsequent work.

[7]See Coleman (1997) and Datta et al. (2002) for the perfect commitment case.

Finally, we consider the question of conditions under which MPNE are also differentiable. For a function[8] $f : K \times L \to \mathbb{R}$, by Hes we denote its Hessian:

$$Hes(f; c, l) = \begin{vmatrix} f^{(1,1)}(c,l) & f^{(2,1)}(c,l) \\ f^{(1,2)}(c,l) & f^{(2,2)}(c,l) \end{vmatrix},$$

and for function $u : K \times L \to \mathbb{R}$ and $g : K \times L \to \mathbb{R}$, we introduce some additional notation as follows:

$$W(u, g; c, l) := u^{(1,1)}(c,l)g^{(2,2)}(s-c, 1-l) + u^{(2,2)}(c,l)g^{(1,1)}(s-c, 1-l)$$
$$-2u^{(1,2)}(c,l)g^{(1,2)}(s-c, 1-l).$$

We now prove our next corollary to our existence theorem that concerns the smoothness of MPNE under some additional conditions that typically hold in most applications:

Corollary 2 (Differentiable MPNE). *Let assumptions 1 and 4 be satisfied with K compact. Assume additionally that $m = 1$ (here we write λ, ρ and g without their indexes for short)*

- $u^{(1)}(0, l) = u^{(2)}(c, 0) = \infty$ *for* $(c, l) \in A(s)$,
- $g^{(1)}(0, l) = g^{(2)}(c, 0) = \infty$ *for* $(c, l) \in A(s)$,
- $Hes(u; c, l) \geq 0$ *and* $Hes(g; c, l) \geq 0$ *where at least one inequality is strict for all* $(c, l) \in A(s)$, *and* $W(u, g; c, l) \geq 0$,
- *For each (Borel measurable, and bounded)* f, $s \to \int_K f(s')\lambda(ds'|s)$ *is differentiable.*

Then, for all h, the best response map $BR(h)(\cdot) \in \mathcal{C}^1(K)$ (and, hence, MPNE are $\mathcal{C}^1$) on the interior of K.

As Amir (1996b) gives conditions under which MPNE are differentiable in a bequest model without elastic labor supply choice, this result extends his result to models with both capital and elastic labor supply (albeit imposing

[8]For every multivariable real valued function $f^{(i,j)}$ is denoted as mixed derivative with respect to i th and j th variable.

stronger conditions than needed for the inelastic labor case). A key difference between our and Amir's method of showing a smooth equilibrium existence is that is we require compact state space K. Differentiability of MPNE is important for numerous reasons. First, its important when seeking to obtain uniform bounds on approximation methods (as it implies the MPNE is locally Lipschitzian.). Second, as observed by Amir (1996b), MPNE differentiability is an important result from of point of view of a Kohlberg (1976) uniqueness result.[9] Finally, concerning the additional assumptions needed in the corollary to show differentiability of MPNE, note that $W(u, g; c, l) \geq 0$ whenever $u(c, l) = u_1(c) + u_2(l)$. The last condition in the hypotheses for $\mathcal{C}^1$ MPNE in the corollary are satisfied, for example, whenever $\lambda(A|s) := \int\limits_A \rho(s, s')\mu(ds')$ for some finite measure μ with $\rho(\cdot, s) \in \mathcal{C}^{(1)}$ and $\int\limits_K \sup\limits_{s \in K} \rho^{(1)}(s, s')\mu(ds') < \infty$. Next, we consider conditions for the existence of monotone MPNE.

Corollary 3 (Monotone MPNE). *Let assumptions 1, 5, as well as the hypotheses (i)-(iii) of Corollary 1 be satisfied. Assume additionally that the kernel $\lambda(\cdot|s)$ is stochastically decreasing, the measure μ is nonatomic, and K is compact. Then, there exists a MPNE (c^*, l^*) with both $c^*(\cdot)$ and $l^*(\cdot)$ increasing and continuous functions.*

Combining results of theorem 1, corollaries 1 and 3, we obtain existence of a continuous and monotone MPNE under very general (complementarity) conditions. Also, notice that although our model is more general than found in the existing literature, our results provide a *weaker* characterization of MPNE than obtained in Amir (1996b) and Nowak (2006) for models with inelastic labor supply (i.e., in models with elastic labor, MPNE policies are not guaranteed to be Lipschitz continuous MPNE). To make clear these differences, we provide Example 3 below, where MPNE exist, but are neither

[9] We can now also add another motivation for showing such a result. Namely, the differentiability of a MPNE allows one to extend a price decentralization of a MPNE allocation is our economies using methods proposed by Lane and Mitra (1981) and extended by Lane and Leininger (1986). What remains to be done, is to decentralize allocations of our stochastic transition. Some steps in this direction has been already taken by Magill and Quinzii (2009). It is important to note that our stochastic production environment differs from theirs along many important dimensions. Noting those differences, it bears mentioning that we are pursuing the question of decentralization of MPNE as a recursive competitive equilibrium for our environment in current work.

monotone nor Lipschitz continuous. In this sense, we conclude that for the case of stochastic altruistic growth with elastic labor, Lipschitzian MPNE cannot be generally expected. It also bears mentioning that we cannot drop Assumption 5 in corollary 3 concerning the existence of monotone Markov equilibrium. More on this example in a moment.

3.2. Uniqueness and Approximation in Models with an Absorbing State

To complete our characterization of MPNE in our baseline model, we now address the question of approximating MPNE in this game. In particular, we are concerned both with iterative methods for computing MPNE, as well as the question of error bounds for standard approximation schemes.

In Theorem 2, we prove an important result concerning the approximation of a MPNE. To do this, we use a simple truncation/iteration argument. One can think of this as studying the structure of pointwise limits of finite iterations (and, hence, the result can be related to finite horizon truncations of our economies). In particular, for $n \geq 1$, and given $s \in K$, we can recursively construct two sequences: $\phi_{2n}(s) = BR(\phi_{2n-1})(s)$, $\phi_{2n+1}(s) = BR(\phi_{2n})(s)$ with $(\forall s \in K)\, \phi_1(s) = (0, 0)$. Similarly we let $\psi_{2n}(s) = BR(\psi_{2n-1})(s)$, $\psi_{2n+1}(s) = BR(\psi_{2n})(s)$ with $(\forall s \in K)\, \psi_1(s) = (s, 1)$. Observe, this can be done as under our assumptions, BR is a function (see lemma 9). With this investment in notation, we present first an existence result per fixed edges (ϕ^d, ϕ^u) and (ψ^d, ψ^u) with a (pointwise) approximation result for a set of MPNE (where the set of fixed edges provide immediate bounds for the actual MPNE set)[10].

Theorem 2 (Approximation of MPNE set). *Let assumptions 1 and 4 be satisfied. Then, the following holds:*

i) there exist limits

$$(\forall s \in K) \quad \phi^d(s) = \lim_{n\to\infty} \phi_{2n-1}(s) \text{ and } \phi^u(s) = \lim_{n\to\infty} \phi_{2n}(s), \tag{1}$$

[10]For a partially ordered set X, for an decreasing operator $A : X \to X$, we say a pair of elements $a \in X$ and $b \in X$ are *fixed edges* of A iff (i) $a \leq b$, and (ii) $A(a) = b$, and $A(b) = a$ (i.e., the pair (a, b) is an ordered 2-cycle of the operator $A(x)$). Therefore, a fixed point of A is a fixed edge but a fixed edge is, of course, not necessarily a fixed point. For example, for $X = [0, 1]$, $A(x) = 1 - x$. For every $a \in X$, $(a, A(a))$ is a fixed edge, but there is a unique fixed point at $x^* = \frac{1}{2}$.

ii) as well as

$$(\forall s \in K) \quad \psi^u(s) = \lim_{n\to\infty} \psi_{2n-1}(s) \text{ and } \psi^d(s) = \lim_{n\to\infty} \psi_{2n}(s), \tag{2}$$

iii) $\phi^u = BR\left(\phi^d\right)$, $\phi^d = BR\left(\phi^u\right)$, *and* $\psi^u = BR\left(\psi^d\right)$, $\psi^d = BR\left(\psi^u\right)$,

iv) if h^* *is a MPNE then* $(\forall s \in K) \quad \phi^d(s) \le h^*(s) \le \psi^u(s)$,

v) if $\phi^d(s) = \psi^u(s)$ *for all* $s \in K$, *there is a unique MPNE* h^*. *Moreover, we have* $h^*(s) = \phi^d(s) = \psi^u(s) = \phi^u(s) = \psi^d(s)$.

We now make a few comments on these computational results. The existence of fixed edges for iterations, as well as the limiting results in the theorem, follow directly from the monotonicity of a BR operator in the model with an absorbing state (see lemma 9). It turns out one direct way of establishing these results is to adapt the methods in (Guo et al. (2004), chapter 3.2) to our problem, where the computation of MPNE takes place with *decreasing* operators. The reasoning behind theorem 2 is straightforward. As the fixed point operator is decreasing, we can define two sequences iterating on the operator from above and below of the strategy space. Although the orbits of the operators themselves are not monotone, it turns out the odd and even elements of the sequences are monotone and hence (pointwise) convergent (sub)sequences (2.(i-ii)).[11] The limit points of all four sequences are fixed edges of operator BR (2.(iii)). Then, the extremal fixed edges are by construction (pointwise) bounding functions for the actual fixed point set. Therefore, Theorem 2.(iv) states our first approximation result, i.e. pointwise bounds for a set of MPNE; then, Theorem 2.(v) provides a type of numerical stability result for iterative methods.

Remark 2. Firstly observe that, unless equal, ϕ^d and ψ^u are not MPNE. Both are however bounding MPNE set. This is different from Balbus et al. (2010) results where BR map is increasing. Secondly, unless $\phi^d = \psi^u$ uniqueness of MPNE cannot be guaranteed. Potential multiple equilibria must be unordered, however (see theorem 1). Thirdly, it is not generally true that

[11] Recall, if for a partially ordered set X, if an operator $A : X \to X$ is decreasing, then the "square" of the operator (i.e., the operator defined as its second orbit, namely $B(x) = A(A(x))$), is increasing. This fact is useful in understanding the nature of our iterative procedures, as well as the computation of extremal fixed edges.

any selection from the interval $[\phi^d, \psi^u]$ is a MPNE. This is a starking difference from the correspondence-based APS methods where any selection from the fixed point correspondence induces a sequential equilibrium (Feng et al., 2009).

To obtain a further characterization of the set of MPNE, we need sufficient conditions on primitives that provide equilibrium uniqueness. These conditions involve the separability of utility with respect to consumption and leisure, as well as an Inada type assumptions on function g. We should mention, these assumptions are often satisfied in models studied in the applied macroeconomics literature.

Theorem 3 (Uniqueness of a MPNE). *Let assumption 2 and 5 be satisfied. Assume additionally $g_1'(0) = g_2'(0) = \infty$. Finally, assume that there exists a number $\tau \in (0,1)$ such that $\forall (c,l) \in Int(A(s))$ (with $s > 0$), we have:*

$$-\frac{\frac{v^{(1)}(c,l)}{v(c,l)}}{\frac{u_1''(c)}{u_1'(c)} + \frac{g_1''(s-c)}{g_1'(s-c)}} - \frac{\frac{v^{(2)}(c,l)}{v(c,l)}}{\frac{u_2''(l)}{u_2'(l)} + \frac{g_2''(1-l)}{g_2'(1-l)}} \leq \tau. \tag{3}$$

Then,

i) there exists a unique MPNE h^* in D. Moreover, if $h_0 \in D$ is an arbitrary starting point in the sequence of iterations $\varphi_{n+1} = BR(\varphi_n)$ with $\varphi_1 = h_0$, and we define $p_n(s) = \int_K v(\varphi_n^1(s'), \varphi_n^2(s'))\lambda(ds'|s)$, then

$$\lim_{n\to\infty} ||p_n - p^*|| = 0, \text{ and } ||p_n - p^*|| \leq M(1 - \tau^{r^n}), \tag{4}$$

where M, τ are constants that depend on a choice of h_0.

ii) $\varphi_n \to h^*$ pointwise,

iii) If additionally K is bounded, λ has Strong Feller Property[12], and the starting point is either $(0,0)$ or $(s,1)$, then

$$\lim_{n\to\infty} ||\varphi_n - h^*|| = 0. \tag{5}$$

[12]By the Strong Feller property, we mean the following: the mapping $s \to \int_K f(s')\lambda(ds'|s)$ is continuous whenever f is bounded and measurable.

Remark 3. Note that the statement iii) in Theorem 3 is satisfied whenever $\varphi_2 \geq \varphi_1$ or $\varphi_2 \leq \varphi_1$ i.e. whenever first iteration is comparable with the starting point.

We make a number of remarks about Theorem 3. First, the theorem gives explicit conditions under which unique MPNE exist in that class D. These conditions, although restrictive, are actually often met in applications. We shall provide examples of this in a moment. Additionally, combining this result with corollaries 1 and 3, we also obtain conditions for the existence of a unique, continuous and monotone MPNE.

Second, our methods for proving uniqueness result are based upon geometric considerations associated with convex operators defined an normal and solid cones. In particular, our result is obtained by showing that under condition (3), an operator corresponding to the current generations best response map is decreasing and *e-convex* (in the terminology of Guo and Lakshmikantham (1988)). Hence, by applying theorem 3.2.5 in Guo et al. (2004), one has a unique fixed point and convergence and approximation results in Theorem 2.(v) follow.

Third, although in Theorem 2, we obtain approximation results for pointwise limits, in our context, we can also obtain uniform convergence results. These bounds follow from condition (3), which guarantees that the cone of functions D in not only normal, but also regular (see Guo et al. (2004) for discussion). This fact turns out to be particularly useful when discussing the nature of error bounds for numerical implementations of our results.

Fourth, although it is difficult to check in our context whether the operator used in the proof of theorem in a contraction, by an important converse to the contraction mapping theorem (see e.g. Leader (1982)), one obtains the fact that it is a contraction (under some suitable metric) indirectly. This argument can be made explicit using exactly the same argument in Balbus et al. (2009) adapted to our setting. This fact is important as it can provide access to additional computation procedures (as well as sharp uniform error bounds) for various standard numerical approximation schemes (e.g., step function approximation) that can be used to compute our unique MPNE.

Finally, the operator used in the proof of this theorem is defined on the set of bounded measurable functions on K, and assigns for any continuation expected utility of the next generation the best response expected utility of a current generation (see proof of theorem 3 for the details). Hence, this operator is an operator defined on the space of *value functions*, and whose construction can be directly (and equivalently) related to correspondence-based strategic dynamic programming methods as discussed in Kydland and Prescott (1980) and Abreu et al. (1990). The key difference per our methods is that our "promised utility" method is adapted to stochastic OLG models without discounting. Further, the value functions associated with the set of MPNE that we construct can be shown to induce a strategic dynamic programming approach defined in spaces of (measurable) correspondences of continuation value functions. In this sense, we have proven that if we restrict our attention to strategic dynamic programming methods that select measurable continuation structures (ala Sleet (1998)) for our environment, this mapping would produce iterations that are described in Theorem 2. That is, in some cases, strategic dynamic programming methods (at least local to a greatest fixed point) can possess geometric structure. Additionally, the way we calculate strategies associated with a particular value function is based on the (generalized) inverse procedure proposed by Coleman (2000). This indicates how all these methods can be unified in the context of our stochastic OLG model without commitment under additional conditions.

We can now provide nice approximation result for MPNE as the limit of finite horizon economies:

Corollary 4 (Finite horizon approximation). *Consider a finite horizon version of our economy, i.e. for an economy of length T, $\forall t < T$, preferences are given by $u_1(c_t) + u_2(l_t) + g(s_t - c_t, 1 - l_t) \int_K v(c_{t+1}(y), l_{t+1}(y))\lambda(dy|s_t)$, and for the last generation, they have utility $u_1(c_T)+u_2(l_T)$. Let assumptions of theorem 3 be satisfied and denote by $h_T := (h^1_T, h^2_T)$ the unique perfect equilibrium strategy of the first generation in T horizon game. Let $p^*_T(s) = \int\limits_K v(h^1_T(s'), h^2_T(s'))\lambda(ds'|s)$. Then,*

$$\lim_{T\to\infty} ||p^*_T - p^*|| = 0,$$

where $p^*(s) = \int\limits_K v(h^*(s'))\lambda(ds'|s)$, and $h^*(s)$ is the unique MPNE from theorem 3. If λ has a Strong Feller Property and K is bounded by Theorem 3 we obtain

$$\lim_{T\to\infty} ||h^T - h^*|| = 0.$$

We should mention, additive separability in consumption and leisure in first period utility is a typical assumption in applied lifecycle models. Further, in Balbus et al. (2009), the authors show that their uniqueness condition (with inelastic labor supply), similar to our condition (3), can be expressed in terms of elasticities of u', g' and v. In particular, by multiplying the numerator and the denominator in inequality (3) by c, we obtain the corresponding "elasticities" interpretation for our uniqueness condition.

To facilitate the understanding of when our uniqueness conditions apply, we now construct a simple example.

Example 1. *Let $u_1(c) = c^{\alpha_1}$, $u_2(l) = l^{\alpha_2}$ and $v(c,l) = c^{\beta_1} l^{\beta_2}$. We find parameters α_i,β_i such that this model satisfies condition 3:*

$$-\frac{\frac{v^{(1)}(c,l)}{v(c,l)}}{\frac{u_1''(c)}{u_1'(c)} + \frac{g_1''(s-c)}{g_1'(s-c)}} - \frac{\frac{v^{(2)}(c,l)}{v(c,l)}}{\frac{u_2''(c)}{u_2'(c)} + \frac{g_2''(1-l)}{g_2'(1-l)}} =$$

$$= \frac{\beta_1}{1-\alpha_1 - \frac{g_1^{''}(s-c)}{g_1^{'}(s-c)}} + \frac{\beta_2}{1-\alpha_2 - \frac{g_2^{''}(1-l)}{g_2^{'}(1-l)}} \leq$$

$$\leq \frac{\beta_1}{1-\alpha_1} + \frac{\beta_2}{1-\alpha_2}.$$

Hence, condition of theorem 3 is satisfied if $\frac{\beta_1}{1-\alpha_1} + \frac{\beta_2}{1-\alpha_2} < 1$ and g is arbitrary function satisfying assumption 4 and conditions of theorem 3.

3.3. Existence in Models without an Absorbing State

Assumption on existence of an absorbing state may be restrictive, especially when the state space K is bounded (see discussion in Balbus et al. (2009)). For this reason, we now state our MPNE existence result for a model without absorbing point. Under assumptions 1 and 3, for a given strategy $h \in D$, the objective becomes now:

$$U(c,l;h,s) := u(c,l) + \beta(h,s)g(s-c,1-l) + \gamma(h,s),$$

with

$$\gamma(h,s) := \int_K v(h_1(y),h_2(y))\lambda_2(dy|s) \quad \text{and}$$

$$\beta(h,s) := \int_K v(h_1(y),h_2(y))\lambda_1(dy|s) - \int_K v(h_1(y),h_2(y))\lambda_2(dy|s).$$

We state the following theorem.

Theorem 4 (Existence of a continuous MPNE). *Under assumptions 1 and 3 there exists a MPNE. If in addition*

i) there exists a μ-measurable function $\bar{\rho}$ such that $\rho_j(s', s) \leq \bar{\rho}(s')$ for each $s', s \in K$ and $j = 1, ..., m$

ii) for each function $f : K \to K$ such that $\int\limits_K f(s')\bar{\rho}(s')\mu(ds') < \infty$ the integral $\int\limits_K f(s')\rho(s', s)\mu(ds')$ is continous as a function of s,

iii) $\int\limits_K v(s', 1)\bar{\rho}(s')\mu(ds') < \infty$.

Then $MPNE = (c^*, l^*)$, where $c^*(\cdot)$ and $l^*(\cdot)$ are continuous functions.

To better understand our results, we now present two additional examples showing application for results 1 and 2. For the moment, assume the bounded state space case (i.e., $K = [0, S]$, where $S \in \mathbb{R}_+$).

Example 2. *In this example, Assumptions 1 and 4 are satisfied*[13]*. Let $u(c, l) = \sqrt{s}\sqrt[4]{l}$ and $v(c, l) = \sqrt{cl}$, and*

$$Q(\cdot|s - c, 1 - l, s) = \sqrt{s - c}\sqrt[4]{1 - l}\lambda(\cdot) + \left(1 - \sqrt{s - c}\sqrt[4]{1 - l}\right)\delta_0(\cdot),$$

where, λ is a uniform distribution on $[0, 1]$. Then, we have

$$U(c, l; h, s) = \sqrt{c}\sqrt[4]{l} + \xi(h)\sqrt{s - c}\sqrt[4]{1 - l},$$

with $\xi(h) := \int\limits_K \sqrt{(h_1(s'))(h_2(s'))}\lambda(ds')$. The best response map BR is a well defined function, and can be described by:

$$BR(h)(s) := \left(\frac{s}{1 + \xi^4(h)}, \frac{1}{1 + \xi^4(h)}\right).$$

[13] Apart from strict concavity of a utility for $l = 0$ and $c = 0$. But this is irrelevant for results of this example as we can easily extend BR to be a function for $s = 0$ since labor supply is constant is this case.

By Theorem 2, we conclude that each perfect equilibrium has the lower bound ϕ^d and upper bound ψ^u, where

$$\phi^d(s) := \lim_{n\to\infty} \phi_{2n-1} \text{ and } \psi^u(s) := \lim_{n\to\infty} \psi_{2n-1},$$

where $\phi_1(s) = (0,0)$ and for $n > 1$

$$\phi_{n+1}(s) = \left(\frac{s}{1+\xi^4(\phi_n)}, \frac{1}{1+\xi^4(\phi_n)} \right).$$

At the same time $\psi_1(s) = (s,1)$ and for $n > 1$

$$\psi_{n+1}(s) = \left(\frac{s}{1+\xi^4(\psi_n)}, \frac{1}{1+\xi^4(\psi_n)} \right).$$

We can compute $\xi(\phi_n)$ by the following recursive formula: $\xi(\phi_1) = 0$; for $n > 1$, we have

$$\xi(\phi_{n+1}) = \frac{\int_K \sqrt{s}ds}{1+\xi^4(\phi_n)} = \frac{\frac{2}{3}}{1+\xi^4(\phi_n)}.$$

The same recursive formula is satisfied for $\xi(\psi_n)$. Only initial value is different, i.e., $\xi(\psi_1) = \frac{2}{3}$. Note, the function $f(x) = \frac{\frac{2}{3}}{1+x^4}$ is decreasing, hence $\xi(\phi_n)$ and $\xi(\psi_n)$ have at most two cumulation points. Both of them must be fixed point of

$$f(f(x)) = \frac{\frac{2}{3}}{1+\left(\frac{\frac{2}{3}}{1+x^4}\right)^4}.$$

Since $f(f(x))$ has exactly one fixed point $\xi^* \approx 0,5932$, this is also the unique fixed point of f. Hence, $\xi(\phi_n) \to \xi^*$ and $\xi(\psi_n) \to \xi^*$. Further, the functions ϕ^d and ψ^u from Theorem 2 are equal and

$$\phi^d(s) = \psi^u(s) = \left(\frac{s}{1+(\xi^*)^4}, \frac{1}{1+(\xi^*)^4} \right) \approx \left(\frac{s}{1.12}, \frac{1}{1.12} \right) \approx (0.89\,s, 0.89).$$

Then, by Theorem 2, strategy above is a unique MPNE. Observe, however, the conditions of Theorem 3 are not satisfied by this example. On the other hand, we show an application of approximation from Theorem 2.

In the next example, we show that strict concavity assumptions on u and g (on the interior of their domain) are needed to guarantee that the mapping BR is a function. This is important when developing constructive procedures for characterizing MPNE.

Example 3. *Let $u(c,l) = \sqrt{cl}$, $v(c,l) = 3\sqrt{cl}$. Transition probability is of the form*

$$Q(\cdot|s-c, 1-l, s) = \sqrt{(s-c)(1-l)}\lambda(\cdot) + \left(1 - \sqrt{(s-c)(1-l)}\right)\delta_0(\cdot),$$

where λ is a uniform distribution on $[0,1]$. Note, neither u nor g is strictly concave in the interior of any $A(s)$, since both functions are linear on the diagonals $d(s) := \{(c,l) : c = ls\}$. We have

$$U(c,l;h,s) = \sqrt{cl} + \xi(h)\sqrt{(s-c)(1-l)},$$

with $\xi(h) := 3\int_K \sqrt{(h_1(s'))(h_2(s'))}\lambda(ds')$. Then, the best response map $BR : D \to 2^D$ is a multifunction described by:

$$BR(h)(s) = \begin{cases} \{(1,s)\} & \text{if } \xi(h) < 1, \\ \{(sl,l) : l \in [0,1]\} & \text{if } \xi(h) = 1, \\ \{(0,0)\} & \text{if } \xi(h) > 1. \end{cases}$$

Hence, the maximal best response (pointwise order) is

$$\overline{BR}(h)(s) = \begin{cases} (1,s) & \text{if } \xi(h) \leq 1, \\ (0,0) & \text{if } \xi(h) > 1. \end{cases}$$

while, the minimal best response is

$$\underline{BR}(h)(s) = \begin{cases} (1,s) & \text{if } \xi(h) < 1, \\ (0,0) & \text{if } \xi(h) \geq 1. \end{cases}$$

Each strategy of the form $h^*(s) = (s\,l^*(s), l^*(s))$ such that $\xi(h^*) = 1$ is a MPNE. This means $l^* : K \to [0,1]$ is arbitrary Borel-measurable function satisfying:

$$\xi(h) = 3\int_S \sqrt{s'}l^*(s')ds' = 1,$$

or, equivalently,

$$\int_S \sqrt{s'} l^*(s')ds' = \frac{1}{3}.$$

Hence, there are many examples of perfect equilibria, for example: $h^1(s) = \left(\frac{s}{2}, \frac{1}{2}\right)$, $h^2(s) = \left(\frac{5}{6}s^2, \frac{5}{6}s\right)$ *and* $h^3(s) = \left(\frac{5}{6}s\sqrt{s(1-s)}, \frac{5}{6}\sqrt{s(1-s)}\right)$. *Note that* h^3 *is neither increasing with respect to* s*, neither Lipschitz continuous. Finally, note* $\phi^d(s) = (0,0)$ *and* $\psi^u(s) = (s,1)$*, hence, in this case, our approximation becomes trivial.*

4. Stationary Markov equilibria

We finally consider the question of existence of Stationary Markov equilibrium (SME). Recall, Theorems 1, 3 and 4 guarantee the existence of MPNE in the models with and without absorbing state. In this section, we study the properties of the equilibrium stochastic process induced by MPNE in our game, and find conditions for existence of an associated nontrivial invariant distributions (and, hence, we can verify the existence of nontrivial SME)

Our first theorem concerns the existence of a SME.

Theorem 5 (Existence of invariant distribution). *Let 1 and 3 be satisfied. Assume moreover that* λ_1 *and* λ_2 *have Feller properties and* $c^*(\cdot)$ *and* $l^*(\cdot)$ *are continuous functions on compact* K*. Then, there exists an invariant distribution.*

We next consider the question of stochastic stability of SME (i.e., uniqueness of invariant distributions for associated MPNE). To obtain uniqueness of an invariant distribution (on possibly unbounded state space), we need stronger assumptions on the primitives of the game. Our uniqueness result is as follows:

Theorem 6 (Uniqueness of invariant distribution). *Assume 1 and 3. If additionally* $c^*(\cdot)$ *and* $l^*(\cdot)$ *are continuous functions,* λ_2 *does not depend on* s *(i.e.* $\lambda_2(\cdot|s) \equiv \lambda_2(\cdot)$*) with a dense in itself support*[14] *and* $\sup_{s\in K} g(s,1) < 1$*, then there exists a unique invariant distribution.*

[14] A set A is said to be dense in itself iff for all $a \in A$ $cl(A \setminus \{a\}) = cl(A)$ where $cl(A)$ is a closure of A. In other words does not contains isolated points.

Now consider the following simple example.

Example 4. *Let* $\lambda_1(\cdot|s) = \lambda_2(\cdot|s)$ *and it is a Dirac delta in* $s+1$*. Then* $Q(\{s+1\}|s) = 1$*. Note that the assumption of theorem 6, i.e.* $\lambda_2(\cdot|s)$ *does not depend on* s *is not satisfied. It is easy to notice that the invariant distribution does not exist.*

Having established SME existence/uniqueness results, we next consider a question of approximating SME. Many of these results are related to those recently obtained in Santos and Peralta-Alva (2005) for competitive economies. In this paper, Santos and Peralta-Alva state sufficient conditions under which every sequence of invariant distributions for approximated policies/strategies converge weakly to an invariant distribution associated with an recursive equilibrium policy function. We adapt these results for our collection of stochastic games. In particular, for the compact state space, say $K = [0, 1]$, we can use the same result here for situations where we have continuous MPNE.

Theorem 7. *Let Assumptions of Theorem 5 be satisfied. Let* c_n *and* l_n *be continuous functions and* $c_n \to c$ *and* $l_n \to l$ *uniformly on* K*. Let* μ_n *be an invariant distribution generated by* (c_n, l_n)*. If* μ^* *is a weak limit point of the sequence* μ_n *then* μ^* *is invariant distribution generated by* (c, l)*.*

Similarly, under stronger assumptions (e.g. when g is a contraction) we can obtain that expected values of random variables representing transition for approximate MPNE, constitute a good approximation of the (unique) invariant distribution of the original MPNE (see theorem 5 and 6 in Santos and Peralta-Alva (2005)).

5. Extensions to Dynastic Models with Multiperiod Structure

In this section, we discuss the possible extensions of our results to broader classes of stochastic OLG economies without commitment. In particular, we consider models where the interactions among agents occurs over multiple periods. We first study the situation where agents have altruism over more than immediate successors, and then focus on case where agents live more than a single period. In some cases, our existence results can be extended to such situations, and in others they cannot. In all situations, our existing computational methods can fail. In each situation, we explain exactly how multiperiod interactions complicate our results, and when existence results are available. We also discuss how new monotone methods (namely, mixed-monotone methods) might be able to resolve these issues.

5.1. Extensions to models with Multiperiod Altruism

We first consider an economy with multiperiod altruism (i.e., where agents care about more than their direct successor). For this situation our existence results remain valid, but our constructive methods can fail.

Example 5 (Models with Multiperiod Altruism). *In this economy, each generation lives one period, but derives utility from consumption/leisure of the next two successor generations. Stochastic production is still given by the transition probability Q (as in the benchmark model of section 2). To construct payoffs, say the current generation assumes that the next two successor generations will use measurable pure strategies* $\phi_1(\cdot) = (c_1(\cdot), l_1(\cdot))$ *and* $\phi_2(\cdot) = (c_2(\cdot), l_2(\cdot))$*, respectively. Then, the payoff of the current generation is defined as the following:*

$$u(c,l) + \\ + \int_K \left[v(\phi_1(y)) + \int_K v(\phi_2(z)) Q(dz|y - c_1(y), 1 - l_1(y), y) \right] Q(dy|s - c, 1 - l, s).$$

A stationary MPNE of the game representing such economy is a policy $\phi^*(\cdot) = (c(\cdot), l(\cdot))$ *that is a best response to* $\phi_1 = \phi_2 = \phi^*$*. Then, our techniques developed earlier in the paper can be used to prove existence of MPNE/SME in this economy exactly as before (e.g., as in Theorem 1) . Unfortunately, when inspecting of the optimization problem of a typical generation (and examining the associated Euler equation for optimal solution), it is clear that the monotonicity of the best response operator can be violated. The reason for this violation is the following: although the objective has decreasing differences with* $(c, l; \phi_2)$*, it does not necessarily have increasing or decreasing differences with* $(c, l; \phi_1)$*. That is, under the assumptions on stochastic production Q in the earlier section of the paper, even assuming v is increasing, and* $\phi_1 = (c_1, l_1)$ *are also increasing in y, inspecting the term*

$$\int_K \left[v(c_1(y), l_1(y)) + \int_K v(\phi_2(z)) Q(dz|y - c_1(y), 1 - l_1(y), y) \right] Q(dy|s-c, 1-l, s)$$

reveals that ϕ_1 induces "mixed" comparative statics in today's decisions rules on (c, l). As a result, one cannot expect the approximation of MPNE set results to hold (see theorems 2, 3), without strong additional assumptions (see fixed-points approximation results for mixed-monotone operators in Guo et al. (2004) applied in Balbus et al. (2009)).

5.2. Dynastic Models with Multiperiod-Lived Agents

We next consider economies where agents live more than 1 period, but there is still strategic altruism. There are various ways this situation can arise. In some cases, our methods work, and in others they fail. We first construct a version of a model where agents live multiple periods that works. In this model, the dynasties that have agents that live two periods, with the "grandparents" in their second period of life ("the old") care about the newly born next period (the grandchildren). In such a model, we can still prove existence.[15]

Example 6 (Two period life length economy with one period altruism). *In this economy, a new generation for each dynasty is born every period, and lives for two periods (as young and old). The generation born at t derives utility from its own consumption/leisure over two consecutive periods of life, and also from altruism associated with the consumption/leisure of the young representatives of the generation born in period $t+2$. The transition probability between states (production level) is as in the benchmark model. Therefore, if the current generation born at period t assumes the next generation of the dynasty is born at period $t+2$ and use policy $\phi(\cdot) = (c_3(\cdot), l_3(\cdot))$, their payoff is given by the following formula:*

$$u(c_1, l_1) + \\ + \int_K \left[u(c_2(y), l_2(y))) + \int_K v(\phi(z)) Q(dz | y - c_2(y), 1 - l_2(y), y) \right] Q(dy | s - c_1, 1 - l_1, s).$$

A stationary MPNE of this game, therefore, is a policy $\phi^*(\cdot) = (c_1(\cdot), l_1(\cdot)) = (c_2(\cdot), l_2(\cdot))$ that is a best response to $\phi^*(\cdot) = (c_3(\cdot), l_3(\cdot))$ of the next generation.

[15] One can equivalently interpret this economy as one, where each period t consists of two "subperiods", with a typically generation conducting economic activity over these two subperiods, and then their children are born at period $t+1$, etc.

As in the example with two-period altruism, the existence of MPNE/SME can be established using tools developed earlier in this paper (e.g., Theorem 1). Unfortunately, as in the two period altruism case, the MPNE approximation results would not be expected to hold. The reason for the complication in this model, though, is somewhat different. Specifically, in this model, although the objective has decreasing differences between $(c_1, l_1; \phi)$ *and* $(c_2, l_2; \phi)$, *the objective is not supermodular with* (c_1, l_1, c_2, l_2). *Hence, the monotonicity of the best response operator will in general be violated. Again, the mixed-monotone methods as discussed in Balbus et al. (2009) might be able to be adapted to analyze such economy.*

Now, we provide a second version of this model where the pattern of altruism makes our existence argument fail. In this version of the multiperiod model, again each generation born at period t lives two periods, and possesses altruistic preferences over the young agents born in period $t+1$ when they are old. In this case, our methods fail, and the existence question must be resolved with an appeal to the arguments proposed by Leininger (1986).

Example 7 (Two period OLG bequest economy). *In this economy, we consider a situation where each period a new generation is born, each living for two periods. Therefore, every period representatives of two generations (young and old) are alive. Each generation is born with a bequest* s *left from the old generation. When young, they divide their endowment* s *into two alternative uses: current consumption* c_1 *and investment for the next period* $s - c_1$. *They also make labor supply decisions. The next period, the production/state level* y *is drawn from transition* Q *parameterized by investment and current labor choice. So when old, the agents divide finished output* y *into tomorrow's consumption* c_2 *and bequest to the generation born in period* $t+1$. *Only representatives of the young generation work.*

Say the current generation (just born) assumes that the generation born in period $t+1$ *uses policy* $\phi(\cdot) = (c(\cdot), l(\cdot))$ *when young. Then, the payoff of this generation born in* t *is:*

$$u(c_1, l_1) + \int\limits_K \left[u(c_2(y)) + v(\phi(y - c_2(y))) \right] Q(dy|s - c_1, 1 - l_1, s),$$

where the term under the integral includes both (i) a return to consumption next period when old $(c_2(y))$ *and (ii) the altruistic return from bequest*

$y - c_2(y)$ to the generation born at $t+1$. Observe that now our methods cannot be applied, even to show existence of a MPNE. The reason is simple: given the strategy of the next generation ϕ, the return to bequest $v(\phi(y - c_2))$ is not concave in c_2 even if v is concave in its arguments. This implies the payoff for the current generation is not concave in period two consumption c_2. This creates two immediate problems. First, it creates the so-called "vicious cycle" problem, when formulating the equilibrium existence problem exactly as in Leininger (1986). That is, it becomes difficult to show that the best response operator maps the set of particular functions (concave or upper-semicontinuous) into itself (see discussion in Leininger (1986) or Amir (1996b)). Second, even if one can find a set of functions that get transformed (i.e., the set of upper semi-continuous monotone functions in their arguments), it is difficult to find a monotone operator that transforms this space. In this case, one can still appeal to the topological construction proposed in Leininger (1986) to prove existence; but unlike the two period altruism case (where mixed-monotone methods can be proposed as in Balbus et al. (2009)), constructive methods seem very difficult to develop for these economies.

One final class of the models with multiperiod lived agents occurs when agents live for multiple periods, but face an uncertain lifetime (e.g. Blanchard, 1985). In this situation, we can again map the economy into our existence results, but computational issues are more complicated.

Example 8 (Stochastic life time economy). *This economy is populated by overlapping generations that live multiple periods, but face a stochastic lifetime. That is, assume that any agent survives to the next period with the probability β and dies with probability $1-\beta$. If an agent does not survive, then the next generation is born. If T_1 is the life time of the current generation and T_2 is the life time of its immediate successor, then the utility of the current generation is given by:*

$$u(c) + \sum_{t=2}^{T_1} u(c_t) + \sum_{t=1}^{T_2} v(c_{t+T_1}).$$

When birth-death process is independent, then T_1 and T_2 are independent random variables with geometric distribution $Prob(T_i = k) = (1-\beta)\beta^k$ (for $k \in \mathbb{N}$). The for the current generation, if the continuation strategy for the successor generation is $\phi(\cdot) = (c(\cdot), l(\cdot))$, and future agents representing *current generation is assumed to be stationary, then the current generations expected lifecycle utility is:*

$$u(c,l) + \int_K \tilde{v}(\phi)(y) Q(dy|s-c, 1-l),$$

where

$$\tilde{v}(\phi)(s) = (1-\beta) E_s \left(\sum_{T_1=0}^{\infty} \beta^{T_1} \left(\sum_{\tau=2}^{T_1} u(\phi(s_\tau)) + (1-\beta) \sum_{T_2=0}^{\infty} \sum_{\tau'=1}^{T_2} \beta^{T_2} v(\phi(s_{T_1+\tau'})) \right) \right).$$

By a simple inspection of this payoff, and as discussed in the examples above (excepting example 7), our methods can be used here to prove existence of the MPNE/SME using a topological argument (e.g., Theorem 1), but again, our approximation results may fail.

5.3. Constructive Methods for Models with Multiperiod Structure

Methods developed in this paper can prove MPNE existence in many examples discussed in the previous section. But as mentioned, in all of multiperiod cases, our equilibrium set approximation result does not in general hold. For this reason, in this section, we discuss how we can integrate a promised utility/APS method (in function spaces) to provide direction for how to compute MPNE, and complement our existence result in the multiperiod models above. To keep the discussion focused, we analyze the model in Example 5 in detail, but as will be clear, the same logic applies for other models for which existence of MPNE can be guaranteed.

We begin by considering a space of value functions

$$\mathcal{P} = \{p : K \to \mathbb{R}_+ \text{ s.t. } p \text{ is bounded and measurable}\}.$$

Endow $2^{\mathcal{P}}$ (the set of all subsets of $\mathcal{P}$) by set inclusion order and weak star topology. Then, define an following APS type operator $B : 2^{\mathcal{P}} \to 2^{\mathcal{P}}$:

$$B(W) = \bigcup_{p \in W} \Big\{ p' \in \mathcal{P} : p'(s) = \int_K v(h_p^*(y)) Q(dy|s - c_p^*(s), 1 - l_p^*(s), s), \text{ where}$$
$$(c_p^*(s), l_p^*(s)) =: h_p^*(s) \in \arg\max \{u(c,l) + \int_K p(y) Q(dy|s-c, 1-l, s)\} \Big\}.$$

Observe that the operator B is monotone, and maps complete lattice $2^{\mathcal{P}}$ into itself. Hence, by Tarski fixed point theorem, it has the greatest fixed point

V^*. Moreover, under additional continuity assumptions needed to guarantee existence, the sequence $W_{t+1} = B(W_t)$ is well-defined (with $W_0 = \mathcal{P}$), and generates a decreasing chain from its "top" element W_0 in $2^{\mathcal{P}}$, with the convergence to $V^* \neq \emptyset$. By an obvious modification of an argument in Balbus et al. (2010), it can easily be verified that when $W \subset B(W)$, $B(W) \subset V^*$ (hence, $B(W)$ is "self-generating"), where V^* is a set of all values generated by any Markov Perfect Nash equilibrium (not necessarily stationary). Hence, $\lim_n B^n(W_0) \rightarrow V^*$, and not only is the existence of MPNE established, but also a direct computational procedure can be used to compute the entire set of sustainable MPNE values. This can be seen as the alternative way of bounding the (nonstationary) MPNE in the case where our direct methods fail.

Observe that although similar, operator B is not a "standard" APS operator. That is, B maps between subsets of functions into themselves (and not spaces of correspondences, as in a standard APS method). Moreover, and more importantly, it is defined pointwise and hence allows one analyze the set of short memory (or Markov) equilibria. Specifically, a standard APS operator applied to this example would produce the larger set of values that support sequential equilibria that include those that are not MPNE (even for our bequest game the is populated by short-lived and memoryless agents). We refer the reader interested in further discussion of both methods discussed in this section, as well as the relationship between our procedure in functions spaces vs standard APS methods in Balbus et al. (2010).

6. Applications to other Environments without Commitment

In this section we discuss additional examples of models where our methods are applied to problems arising in macroeconomics, public finance, developmental and environmental economics, including models with human capital accumulation, endogenous fertility, and endogenous environmental quality. For the first two applications, our existence and computation results are valid. In the last applications, we establish the equilibrium existence only. We begin with an application out of macroeconomics.

Example 9 (Human Capital Models without Commitment). *Consider an economy where each generation lives one period but derives utility from consumption and time spend to educate children (of both itself and that of its immediate successor). The state is given by* $s = (k, h)$*, where* $k \in [0, S]$

denotes the level of physical capital and $h \in [0, H]$ *level of human capital. Transition probability between state* s *and* $A \subset [0, \bar{K}] \times [0, \bar{H}] = K$ *is given by* $Q(A|i_k, i_h, s)$*, where* i_k *is the physical capital bequest level, while* i_h *investment in the children education. Let* $f(k, l)$ *be a production function (producing nonstorable consumption good) from physical capital and level* l *of human capital chosen for work. The rest of human capital* $h - l$ *is used to educate children and increase their future human capital. Assuming that the next generation use policy* $\phi^*(\cdot) = (\tilde{c}(\cdot), \tilde{l}(\cdot))$*, the utility of the current generation is given by:*

$$u(c, h-l) + \int_K v(\tilde{c}(y_k, y_h), y_h - \tilde{l}(y_k, y_h))Q(dy|f(k,l) - c, h - l, s).$$

A stationary MPNE of the game representing such economy is a policy $\phi^*(\cdot) = (c(\cdot), l(\cdot))$ *that is a best response to* ϕ^**. Let* $Q(\cdot|f(k,l) - c, h - l, s) = \sum_{i=1}^{m} g_i(f(k,l) - c, h - l)\lambda_i(\cdot|s) + g_0(f(k,l) - c, h - l)\delta_0(\cdot)$ *with* g, f *twice continuously differentiable, increasing, concave and* g *supermodular. Let* u *be separable,* $f(0, l) = 0$ *and assume also Inada type conditions to obtain the interior solution. Hence our methods can be used to obtain MPNE existence, antichain structure of the MPNE set (see theorem 1), uniqueness and computational results under similar assumptions as in theorem 3.*

We now present an example inspired by environmental quality control models of Jouvet et al. (2000), Jones and Manuelli (2001), John and Pecchenino (1994). This is also an example of how our methods can be extended to multisector models.

Example 10 (Two Sector Models of Environmental Quality). *Consider an economy where each generation lives one period but derives utility from own clean and dirty consumption as well as that of its immediate successor. The state is given by* $s = (k_c, k_d)$*, where* $k_c, k_d \in [0, S]$ *denotes the level of clean and dirty capital. Transition probability between state* s *and* $A \subset [0, S] \times [0, S] = K$ *is given by* $Q(A|f_c(k_c, k_d) - c_c, f_d(k_c, k_d) - c_d, s)$*, where* $f_i (i = c, d)$ *is a production function of clean and dirty consumption goods. Production of both kind of goods is separable. Each generation leaves* $f_c(k_c, k_d) - c_c$ *and* $f_d(k_c, k_d) - c_d$ *of clean and dirty bequest to the next generation. Assuming that the next generation use policy* $\phi(\cdot) = (\tilde{c}_c(\cdot), \tilde{c}_d(\cdot))$*, the* *utility of the current generation is given by*

$$u(c_c, c_d) + \int_K v(\tilde{c}_c(y_c, y_d), \tilde{c}_d(y_c, y_d))Q(dy|f_c(k_c, k_d) - c_c, f_d(k_c, k_d) - c_d, s).$$

A stationary MPNE of the game representing such economy is a policy $\phi^*(\cdot) = (c_c(\cdot), c_d(\cdot))$ *that is a best response to* ϕ^**. Assume that* $Q(\cdot|f_c(k_c, k_d) - c_c, f_d(k_c, k_d) - c_d, s) = \sum_{i=1}^{m} g_i(f_c(k_c, k_d) - c_c, f_d(k_c, k_d) - c_d)\lambda_i(\cdot|s) + g_0(f_c(k_c, k_d) - c_c, f_d(k_c, k_d) - c_d)\delta_0(\cdot)$ *with* g, u *continuously differentiable, supermodular, concave and increasing, with* $f(0,0) = 0$*. Observe that the objective is then supermodular with* (c_c, c_d) *and has decreasing differences with* c_i, ϕ*. Hence, our methods can be used to obtain MPNE existence, antichain structure of the MPNE set (see theorem 1), uniqueness and computational results under assumptions of theorem 3.*

In our final example, we discuss an altruistic OLG model with endogenous fertility inspired by de la Croix and Doepke (2004). Unfortunately, for these economies, although our methods verify existence of MPNE, our computation results fail here to compute the MPNE set.

Example 11 (Endogenous fertility Models without Commitment). *Consider an economy where each generation lives one period but derives utility from own consumption and number of children as well as consumption/number of children of their immediate successor. The state (output level) is given by* s*. The fertility is endogenous. It is assumed that raising one child takes* γ *% of parent's time and costs* α *in terms of current consumption good. Parents leave bequest for the next generations* $s - c - \alpha n$ *that together with time devoted to work* $1 - \gamma n$ *parameterize the transition probability between the current states and state in the next period:* $Q(\cdot|\frac{s-c-\alpha n}{n}, 1 - \gamma n, s)$*. It is assumed that every child gets the same draw* y *and behaves identically.*

Assuming that the next generation use policy $\phi(\cdot) = (\tilde{c}(\cdot), \tilde{n}(\cdot))$*, the utility of the current generation is given by*

$$u(c, n) + n \int_S v(\tilde{c}(y), \tilde{n}(y))Q(dy|\frac{s - c - \alpha n}{n}, 1 - \gamma n, s).$$

A stationary MPNE of the game representing such economy is a policy $\phi^*(\cdot) = (c(\cdot), n(\cdot))$ *that is a best response to* ϕ^**. Observe that our method applied to* *this economy can prove existence of the MPNE by continuity of the best response map. The operator is not monotone, however. That is, still we have decreasing differences with $(c;\phi)$ but not in $(n;\phi)$. Moreover, we loose supermodularity of the payoff with (c,n), and hence the lattice structure of the feasible set of actions: $(c,n) \in [0,s] \times [0,\frac{1}{\gamma}]$ with $c + \alpha n \leq s$.*

7. Proofs

In this concluding section of the paper, we present all the proofs of the results in the paper. We begin with the proofs for the model with an absorbing state.

7.1. Proofs in the model with absorbing state

In this section we assume 1 and 4. We start by extending the set of strategies D to a set of randomized policies: $\mathcal{D}$, i.e. if $\bar{h} \in \mathcal{D}$ then $\bar{h}$ is a transition probability from K to $K \times L$ such that $\bar{h}\left(A(s)|s\right) = 1$. Similarly define $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$ the following way

$$\mathcal{BR}\left(\bar{h}\right)(s) := \arg \max_{(c,l) \in A(s)} U(c,l;\bar{h},s),$$

where

$$U(c,l;\bar{h},s) := u(c,l) + \sum_{k=1}^{m} \int_K \int_{K \times L} v(c',l')\bar{h}(dc',dl'|s)\lambda_k(dy|s)g_k(s-c,1-l).$$

Following Nowak (2006) or Balbus and Nowak (2008), we endow $\mathcal{D}$ with the weak-star topology. By a Caratheodory function $w : C \to R$ on $C := K \times A$ with $A := K \times L$, we mean a function w such that $w(s,\cdot)$ is continuous on $A(s)$ for each $s \in K$, $w(\cdot,a)$ is Borel measurable for each $a \in A(s)$, and $s \to \max_{a \in A(s)} |w(s,a)|$ is μ-integrable over K. Since all the sets $A(s)$ are compact, $\mathcal{D}$ is compact and metrizable.[16] We mention that a sequence $\bar{h}_n$ converges to $\bar{h}$ if and only if for every Caratheodory function we have:

$$\int\limits_K \int\limits_{A(s)} w(s,a)\bar{h}_n(da|s)\mu(ds) \to \int\limits_K \int\limits_{A(s)} w(s,a)\bar{h}(da|s)\mu(ds).$$

[16]For the details we refer the reader to Balder (1980) or Chapter IV in Warga (1972).

Observe that $\mathcal{D}$ could be treated as the set of equivalence classes of correlated strategies equal μ a.e. Note that each function $h \in D$ can be treated as a member of $\mathcal{D}$ which has property $\bar{h}(s) \in \{h(s)\}$ μ a.e.

Now define an auxiliary function

$$F(c,l,\xi,s) := u(c,l) + \sum_{k=1}^{m} \xi_k g_k(s-c, 1-l),$$

with $\xi := (\xi_1, ..., \xi_m) \in \mathbb{R}^m_{++}$. We formulate a lemma.

Lemma 8. *Function F is supermodular in (c,l) and has decreasing differences in $(c,l;\xi)$ for any $s \in K$.*

Proof. Note that by Assumption 1 and 4 $(c,l) \to F(c,l,\xi,s)$ is supermodular. Moreover both functions $c \to F(c,l,\xi,s)$ and $l \to F(c,l,\xi,s)$ are strictly concave. Let $\xi^1 \le \xi^2$ in the product order sense. Then by Assumption 4 we have

$$\begin{aligned} F^{(1)}\left(c,l,\xi^1,s\right) &= u^{(1)}(c,l) - \sum_{k=1}^{m} \xi_k^1 g_k^{(1)}(s-c,1-l) \\ &\ge u^{(1)}(c,l) - \sum_{k=1}^{m} \xi_k^2 g_k^{(1)}(s-c,1-l) \\ &= F^{(1)}\left(c,l,\xi^2,s\right). \end{aligned} \tag{6}$$

Hence we easily conclude that F has desired decreasing differences in $(c;\xi)$. We proceed similarly to show that F has decreasing differences in $(l;\xi)$.

Lemma 9. *Function BR is well defined and decreasing.*

Proof. By assumptions 1 and 4 the function $(c,l) \to U(c,l;h,s)$ is strictly concave. Hence there is exactly one solution to the maximizing problem of $(c,l) \to U(c,l;h,s)$. Hence $BR(h)$ is well defined function $BR : D \to D$.

We show monotonicity of BR. Fix $s \in K$. Note that $U(c,l;h,s) = F(c,l;\xi(h),s)$ where k-th coordinate of the vector $\xi(h)$ is

$$\xi_k(h) = \int_K v(h_1(s'), h_2(s'))\lambda_k(ds'|s).$$

Note that by Lemma 8 U is a supermodular and continuous function with (c,l) on lattice $A(s)$ and has decreasing differences with $(c,l;h)$. By Topkis (1978) the (unique) selection $BR(\cdot)(s)$ is decreasing for any s.

PROOF (OF THEOREM 1). First we show that $\mathcal{BR}$ is a well defined function mapping $\mathcal{D}$ to $\mathcal{D}$. Let $\bar{h} \in \mathcal{D}$. Note that

$$U(c,l;\bar{h},s) = F(c,l;\xi(\bar{h},s),s),$$

where

$$\begin{aligned}
\xi_k(\bar{h},s) &:= \int\limits_K \int\limits_{A(s)} v(c',l')\bar{h}(dc',dl'|s')\lambda_k(ds'|s) \\
&= \int\limits_K \int\limits_{A(s)} v(c',l')\rho_k(s',s)\bar{h}(dc',dl'|s')\mu(ds').
\end{aligned}$$

Note that by definition of F, assumption 1 and hence strict concavity of $(c,l) \to U(c,l;h,s)$ on $A(s)$ we immediately obtain that there is a unique optimal solution of maximization problem of $U(c,l;\bar{h},s)$. Hence we have shown that $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$. Moreover by strict concavity, the image of $\mathcal{BR}$ is contained in D i.e. $\mathcal{BR}(\mathcal{D}) \subset D$. Now we show that $\mathcal{BR}$ is continuous in the weak-star topology. Let $\bar{h}_n \to \bar{h}$ in the weak star topology. Note that if $a = (c,l)$ then for each $s \in K$ the function

$$w_k(s',a) := v(a)\rho_k(s',s)$$

is a Caratheodory function. Hence

$$\xi_k(\bar{h}_n,s) \to \xi_k(\bar{h},s) \quad \text{as } n \to \infty,$$

and $U(c,l;\bar{h}_n,s) \to U(c,l;\bar{h},s)$. Since for each $\bar{h}$ and s the function $(c,l) \to U(c,l;\bar{h},s)$ is strictly concave, hence by Berge maximum theorem optimal solution of $U(c,l;\bar{h}_n,s)$ must converge to the optimal solution of $U(c,l;\bar{h},s)$. Hence $\mathcal{BR}(\bar{h}_n) \to \mathcal{BR}(\bar{h})$ pointwise. Since $\mathcal{BR}(\bar{h}_n)$ and $\mathcal{BR}(\bar{h})$ are pure strategies, hence this convergence also holds in the weak star topology. Hence $\mathcal{BR}$ is continuous. $\mathcal{D}$ is compact in this topology, hence by Schauder-Tikhonov theorem we conclude that there exists fixed point $h^* = \mathcal{BR}(h^*)$ μ a.e. Let $h^o(s) := \mathcal{BR}(h^*)(s)$ pointwise. Since $\mathcal{BR} : \mathcal{D} \to D$, hence h^o must be a stationary strategy. Since $h^o = h^*$ μ a.e. by definition of the function $\xi(h,s)$ we conclude that $\xi(h^*,s) = \xi(h^o,s)$ for each $s \in K$ and hence for each (c,l) we have $U(c,l;h^o,s) = U(c,l;h^*,s)$. Hence $h^o = h^*$ for each $s \in K$ and $BR(h^o)(s) = h^o(s) = \mathcal{BR}(h^*)(s)$ for each $s \in K$.

Finally the antichain structure of MPNE set results directly form the fact that BR is decreasing (see lemma 9).

Proof (of Corollary 1). We show that a stationary MPNE $(c^*(s), l^*(s))$ is a continuous function of s. It is sufficient to show that BR maps D into the set of bounded, continuous functions on K. Let $h \in D$. Let $s_n \to s_0$ as n tends to ∞. By condition (iii) of this corollary we have

$$\int_K v(h_1(s'), h_2(s'))\bar{\rho}_j(s')\mu(ds') \;\leq\; \int_K v(s', 1)\bar{\rho}_j(s')\mu(ds') < \infty. \tag{7}$$

Hence and by (ii) we immediately obtain

$$\int_K v(h_1(s'), h_2(s'))\lambda_j(ds'|s_n) \to \int_K v(h_1(s'), h_2(s'))\lambda_j(ds'|s_0). \tag{8}$$

Let $(c_n, l_n) := BR(h)(s_n)$ and (c_0, l_0) be an arbitrary cumulation point of (c_n, l_n).

$$U(c_n, l_n; h, s_n) \geq U(c, l; h, s_n)$$

for all $(c, l) \in A(s)$ and $n \in \mathbb{N}$. Hence and by (8) we immediately obtain

$$U(c_0, l_0; h, s) \geq U(c, l; h, s_0),$$

for all $(c, l) \in A(s)$. Hence $(c_0, l_0) := BR(h)(s_0)$. Since BR maps D into set of continuous functions, hence $(c^*(\cdot), l^*(\cdot))$ must be a continuous function.

Proof (of Corollary 2). We obtain FOC:

$$u^{(1)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g_k^{(1)}(s - c, 1 - l) \;=\; 0, \tag{9}$$

$$u^{(2)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g_k^{(2)}(s - c, 1 - l) \;=\; 0. \tag{10}$$

We need to show that Jacobian of the function $G(c, l) = [G_1(c, l)\, G_2(c, l)]'$ say $J(G)(c, l)$ with

$$G_1(c, l) := u^{(1)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g^{(1)}(s - c, 1 - l), \tag{11}$$

$$G_2(c, l) := u^{(2)}(c, l) - \int_K v(h_1(s'), h_2(s'))\lambda(ds'|s)g^{(2)}(s - c, 1 - l), \tag{12}$$

is not zero. Note that

$$G_i^{(j)}(c,l) = u^{(i,j)}(c,l) + \int_K v(h_1(s'),h_2(s'))\lambda(ds'|s)g_k^{(i,j)}(s-c,1-l),$$

and

$$\begin{aligned}
J(G)(c,l) &:= \prod_{i=1}^{2}\left(u^{(i,i)}(c,l)+\gamma(s)g^{(i,i)}(s-c,1-l)\right)\\
&- \left(u^{(1,2)}(c,l)+\gamma(s)g^{(1,2)}(s-c,1-l)\right)^2,\\
&= u^{(1,1)}(c,l)u^{(2,2)}(c,l)-\left(u^{(1,2)}(c,l)\right)^2\\
&+ \gamma^2(s)\left(g^{(1,1)}(s-c,1-l)g^{(2,2)}(s-c,1-l)-\left(g^{(1,2)}(s-c,1-l)\right)^2\right)\\
&+ \gamma(s)\left(u^{(1,1)}(c,l)g^{(2,2)}(s-c,1-l)\right.\\
&+ u^{(2,2)}(c,l)g^{(1,1)}(s-c,1-l)\\
&- \left.2u^{(1,2)}(c,l)g^{(1,2)}(s-c,1-l)\right),\\
&= Hes(u;c,l)+\gamma^2(s)Hes(g;s-c,1-l)\\
&+ \gamma(s)W(u,g;c,l)>0
\end{aligned}$$

with $\gamma(s) = \int_K v(h_1(s'),h_2(s'))\lambda(ds'|s)$. By implicit function theorem we obtain $BR(h)$ is in $\mathcal{C}^1$ in the interior of K (say $Int(K)$). Hence we show that BR maps D into a set of differentiable functions on $Int(K)$.

Proof (of Corollary 3). Note that the utility has now a form

$$U(c,l;h,s) = u(c,l) + g_1(s-c)\int_K v(h_1(s'),h_2(s'))\lambda(ds'|s)$$

$$+g_2(1-l)\int_K v(h_1(s'),h_2(s'))\lambda(ds'|s).$$

Fix arbitrary $s>0$, and $h \in D$. If $\int_K v(h_1(s'),h_2(s'))\lambda(ds'|s) > 0$, then by Assumption 1 and 5 this function above is strictly concave on $A(s)$, and hence maximization problem of $(c,l)\to U(c,l;h,s)$ has a unique solution. If $\int_K v(h_1(s'),h_2(s'))\lambda(ds'|s) = 0$, then only $(s,1)$ is an optimal solution. Hence the best response map BR is a well defined function.

Finally observe that by assumptions for each $h \in D$ where h is increasing, function $(c, l, s) \to U(c, l, ; h, s)$ is supermodular in (c, l) and has increasing differences in (c, s) and (l, s). Hence by Topkis (1978) theorem for each increasing $h \in D$ function $BR(h)(\cdot)$ in increasing on K.

As a result BR maps increasing and bounded functions into increasing and bounded function. Hence and by Corollary 1 we know that BR maps increasing functions from D into continuous and increasing functions in D. To finish the proof we need to show that BR is continuous in the *weak topology*[17].

Let $\Psi := \{(c, l) \in D : c, l$ are increasing and usc and continuous in $s.\}$. Observe, $BR(\Psi) \subset \Psi$. Endow Ψ with a product of weak topologies. We show that Ψ is a compact set. Denote $\Delta(K) \times \Delta(K)$ as a Cartesian product of probability measures on K endowed with a weak topology. Let us define a function $\mathcal{T} : \Psi \to \Delta(K) \times \Delta(K)$, as follows: $\mathcal{T}(c, l) = (\tilde{c}, \tilde{l})$ i.e. a pair of probability measures with distribution functions $\tilde{c}(s) = \frac{c(s)}{S}$ for $s < S$, $\tilde{c}(S) = 1$, and $\tilde{l}(s) = l(s)$ for $s < S$ and $\tilde{l}(S) = 1$. Clearly these distributions are supported on K. Define $\Delta_0 := \Delta(K) \times \Delta(K)$. Note that, $\mathcal{T}$ is a homeomorphism between Ψ and Δ_0. Observe that $\Delta(K)$ is tight[18] as a collection of measures with a common compact support. Hence by Prohorov Theorem (see Section 5 in Billingsley (1999)) Δ_0 is compact. Hence Ψ is a compact set. We show that BR restricted to Ψ is continuous in the weak topology. Let $h_n(s) := (c_n(s), l_n(s)) \to (c(s), l(s)) =: h(s)$ weakly. Since $s \to c(s)$ and $s \to l(s)$ are increasing functions, its set of discontinuity points is at most countable. Since by assumption of this corollary μ is nonatomic, hence h must be continuous $\mu-$almost everywhere. As a result: $h_n \to h$ $\mu-$almost everywhere, and also $v(c_n(s), l_n(s)) \to v(c(s), l(s))$ for $\mu-$almost all s. Since $v(c_n(s), l_n(s)) \leq v(S, 1) < \infty$ hence by Lebesgue Dominance Theorem

$$\int_K v(c_n(s'), l_n(s'))\rho(s, s')\mu(ds') \to \int_K v(c(s'), l(s'))\rho(s, s')\mu(ds'),$$

for every s. This implies that $U(c, l, h_n, s) \to U(c, l, h, s)$. Since for each h and s the function $(c, l) \to U(c, l; h, s)$ is strictly concave, hence by Berge

[17] A sequence of the functions $f_n \to f$ weakly iff $f_n(s) \to f(s)$ for all continuity points of f.

[18] Collection $\mathcal{M}$ of probability distributions is tight iff for every $\epsilon > 0$ there is compact set K such that $\mu(A) \geq 1 - \epsilon$ for all $\mu \in \mathcal{M}$.

maximum theorem optimal solution of $U(c,l;h_n,s)$ must converge to the optimal solution of $U(c,l;h,s)$ for all s. Hence $BR(h_n) \to BR(h)$ pointwise, hence also in the weak topology. Therefore BR restricted to Ψ is continuous in the weak topology. Hence there is continuous and increasing fixed point of BR as $BR(\Psi)$ is contained in the set of continuous and increasing functions.

Now we state an auxiliary lemma:

Lemma 10. *BR is a pointwise continuous function i.e. if $h_n \to h$ pointwise, then $BR(h_n)(\cdot) \to BR(h)(\cdot)$ pointwise as well.*

PROOF. Since $(c,l) \to U(c,l;h,s)$ is strictly concave and continuous on $A(s)$ it is sufficient to show that $h \to U(c,l;h,s)$ is continuous. Note that:

$$U(c,l;h,s) = F(c,l,\xi(h,s),s).$$

Clearly F is continuous in ξ. It is sufficient to show that $\xi(\cdot,s)$ is continuous in the pointwise topology. If $h_n \to h$ pointwise we obtain $\xi(h_n,s) \to \xi(h,s)$ by Assumption 1,4 and Lebesgue Dominance Theorem. Hence $U(c,l;h,s)$ is continuous in h as superposition of continuous functions. Using Berge maximum theorem the convergence holds pointwise.

PROOF (OF THEOREM 2). *Step 1.* We prove (i). We show that ϕ_{2n-1} is increasing and ϕ_{2n} is decreasing. Clearly $\phi_1 \le \phi_3$ and $\phi_1 \le \phi_2$. By Lemma 9 and definition of sequence ϕ_n we obtain:

$$\phi_2 = BR(\phi_1) \ge BR(\phi_3) = \phi_4.$$

Suppose that for some n $\phi_{2n} \ge \phi_{2(n+1)}$ and $\phi_{2n-1} \le \phi_{2n+1}$ hold. By Lemma 9 and definition of sequence ϕ_n we obtain:

$$\phi_{2n+1} = BR(\phi_{2n}) \le BR(\phi_{2n+2}) = \phi_{2n+3}.$$

Therefore

$$\phi_{2(n+2)} = BR(\phi_{2n+3}) \le BR(\phi_{2n+1}) = \phi_{2(n+1)}.$$

Finally we obtain that both sequences ϕ_{2n} and ϕ_{2n-1} are monotone and bounded. Hence the limits in (1) exist.

Step 2. We prove (ii). We show that ψ_{2n-1} is decreasing and ψ_{2n} is increasing. Clearly $\psi_1 \ge \psi_3$ and $\psi_1 \ge \psi_2$. By Lemma 9 and definition of sequence ψ_n we obtain:

$$\psi_2 = BR(\psi_1) \le BR(\psi_3) = \psi_4.$$

Suppose that for some n $\psi_{2n} \leq \psi_{2(n+1)}$ and $\psi_{2n-1} \geq \psi_{2n+1}$ hold. By Lemma 9 and definition of sequence ψ_n we obtain:

$$\psi_{2n+1} = BR\left(\psi_{2n}\right) \geq BR\left(\psi_{2n+2}\right) = \psi_{2n+3}.$$

Therefore

$$\psi_{2(n+2)} = BR\left(\psi_{2n+3}\right) \geq BR\left(\psi_{2n+1}\right) = \psi_{2(n+1)}.$$

Finally we obtain that both sequences ψ_{2n} and ψ_{2n-1} are monotone and bounded. The limits, hence, exist.

Step 3. We prove (iii). Note that $\phi_{2n+1} = BR(\phi_{2n})$. By Step 1 and by Lemma 10 we obtain $\phi^u = BR\left(\phi^d\right)$. By analogue reasoning we obtain the rest of results.

Step 4. We prove (iv). By definition of h^* we know that $h^* = BR\left(h^*\right)$. By definition of BR and ϕ_n and ψ_n we immediately obtain

$$\phi_1 \leq BR(h^*) = h^* \leq \psi_1. \tag{13}$$

Assume that for some n:

$$\phi_{2n-1} \leq h^* \leq \psi_{2n-1}. \tag{14}$$

Note that:

$$\phi_{2n+1} = BR\left(BR(\phi_{2n-1})\right) \quad \text{and} \quad \psi_{2n+1} = BR\left(BR(\psi_{2n-1})\right). \tag{15}$$

Observe that by lemma 9 function $BR \circ BR(\cdot)$ is increasing. Hence, combining (13), (14), (15) we obtain:

$$\begin{aligned}
\psi_{2n+1} &= BR\left(BR(\psi_{2n-1})\right) \geq BR\left(BR(h^*)\right) \\
&= h^* \\
&= BR\left(BR(h^*)\right) \geq BR\left(BR(\phi_{2n-1})\right) \\
&= \phi_{2n+3}. \qquad (16)
\end{aligned}$$

To finish the proof we just take a limit in (16).

Step 5. Proof of (v) is immediate from Theorem 1 and from (iv).

7.2. Proofs in the model with separated utility variables and absorbing state

By assumptions of Theorem 3 the objective becomes:

$$U(c,l;h,s) := u_1(c) + u_2(l) + \xi(h,s)[g_1(s-c) + g_2(1-l)],$$

with $\xi(h,s) := \int\limits_K v(h_1(y), h_2(y))\lambda(dy|s)$.

Proof (of Theorem 3). *Step 1* We prove *i)* Let $\mathcal{P}$ be a set of bounded, Borel measurable functions $p : K \to \mathbb{R}_+$with a pointwise partial order and the topology of uniform convergence. Clearly $\mathcal{P}$ is a normal solid cone. Define an operator $T : \mathcal{P} \to \mathcal{P}$:

$$T(p)(s) = \int\limits_K v(c_p(s'), l_p(s'))\lambda(ds'|s),$$

where $(c_p(s), l_p(s))$ is a measurable solution (refer to Brown and Purves (1973) theorem 2) of optimization problem of the function

$$H(c,l;p,s) := u_1(c) + u_2(l) + p(s)(g_1(s-c) + g_2(1-l)).$$

Clearly H has decreasing differences in (c,p) and (l,p), hence by Topkis (1978) theorem (c_p, l_p) is decreasing. By assumption 2 $T(\cdot)$ is decreasing as well.

Now we show that the function $J(t) := t^\tau T(tp)$ $t \in (0,1)$ is increasing for each $p \in Int\,(\mathcal{P})$ and τ from condition (3). Adding continuity of J at $t = 1$ we obtain that $T(tp) \leq t^{-\tau}T(p)$, i.e. the e-convexity condition for T in theorem 3.2.5 of Guo et al. (2004). Fix p from interior of $\mathcal{P}$ and $s \in K \setminus \{0\}$. Define $c(t) := c_{tp}$ and $l(t) := l_{tp}$.

First note that by $u_i'(0^+) = g_i'(0^+) = \infty$ $(i = 1,2)$ for $t \in (0,1)$ we have $(c(t), l(t)) \in Int(A(s))$.

Hence the equalities are satisfied

$$H^{(1)}(c(t), l(t); tp(s), s) = u_1'(c(t)) - tp(s)g_1'(s - c(t)) = 0$$

and

$$H^{(2)}(c(t), l(t); tp(s), s) = u_2'(l(t)) - tp(s)g_2'(1 - l(t)) = 0.$$

By implicit function theorem both $c(t)$ and $l(t)$ are differentiable and:

$$-c'(t) = \frac{p(s)g_1'(s - c(t)}{-\left(u_1''(c(t)) + tp(s)g_1''(s - c(t))\right)},$$

and

$$-l'(t) = \frac{p(s)g_2'(1-l(t))}{-\left(u_2''(l) + tp(s)g_2''(1-l(t))\right)}.$$

Then we have:

$$\begin{aligned}
-\frac{d}{dt}v(c(t),l(t)) &= -v^{(1)}(c(t),l(t))c'(t) - v^{(2)}(c(t),l(t))l'(t), \\
&= v^{(1)}(c(t),l(t))\frac{p(s)g_1'(s-c(t))}{-\left(u_1''(c(t)) + tp(s)g_1''(s-c(t))\right)}, \\
&+ v^{(2)}(c(t),l(t))\frac{p(s)g_2'(1-l(t))}{-\left(u_2''(l(t)) + tp(s)g_2''(1-l(t))\right)}, \\
&= -\frac{1}{t}\frac{\frac{v^{(1)}(c(t),l(t))}{v(c(t),l(t))}}{\frac{u_1''(c(t))}{u_1'(c(t))} + \frac{g_1''(s-c(t))}{g_1'(s-c(t))}}v(c(t),l(t)), \\
&- \frac{1}{t}\frac{\frac{v^{(2)}(c(t),l(t))}{v(c(t),l(t))}}{\frac{u_2''(l(t))}{u_2'(l(t))} + \frac{g_2''(1-l(t))}{g_2'(1-l(t))}}v(c(t),l(t)), \\
&\leq \frac{\tau}{t}v(c(t),l(t)) \leq \frac{\tau}{t}v(s,1). \qquad (17)
\end{aligned}$$

The last inequality follows directly from (3). Hence if $c(t,s) := c(t)$ then the derivative of $v(c(t,s),l(t,s))$ is integrable with respect to probabilistic measure $\lambda(\cdot|s)$ since $v(s,1)$ is integrable by assumption 2. Hence by (17) we obtain:

$$\begin{aligned}
t\frac{d}{dt}\left(\int\limits_K v(c(t,s'),l(t,s'))\lambda(ds'|s)\right) &= t\int\limits_K \frac{d}{dt}v(c(t,s'),l(t,s'))\lambda(ds'|s) \\
&\geq -\tau\int\limits_K v(c(t,s),l(t,s))\lambda(ds'|s) \\
&= -\tau T(tp).
\end{aligned}$$

Therefore,

$$
\begin{aligned}
J'(t) &= t^{\tau-1}\left(\tau T(tp) + t\left(\int\limits_K \frac{\partial}{\partial t} v(c(t,s'), l(t,s'))\lambda(ds'|s)\right)\right) \\
&\geq t^{\tau-1}\left(\tau T(tp) - \tau T(tp)\right) \\
&= 0.
\end{aligned}
$$

Hence $J(t)$ is increasing on $(0,1)$. Since J is continuous, J is decreasing on all $[0,1]$. Hence by Guo et al. (2004) we obtain that T possesses a unique fixed point, say p^*. Moreover, each sequence of iterations $p_{n+1} = T(p_n)$ (with p_0 arbitrary starting point) converges to p^*. Moreover condition (4) holds. Hence there exists a unique $h^* := (c^*, l^*)$ such that $p^*(s) = \int\limits_K v(c^*(s'), l^*(s'))\lambda(ds'|s)$, where the pair $(c^*(s), l^*(s))$ solves optimization problem of the function $(c,l) \to H(c, l; p^*, s)$. Moreover, (c^*, l^*) is a unique perfect equilibrium.

Step 2. Fix $s > 0$. For $(c,l) \in A(s)$ and $\xi > 0$ let $F(c, l; \xi) := u_1(c) + u_2(l) + \xi\left(g_1(s-c) + g_2(1-l)\right)$. Then H is of the form $H(c, l; p, s) = F(c, l; p(s))$. Note that from assumptions 2 and 5 F is continuous as a function of $(c, l; \xi)$. Clearly F is strictly concave on a compact set $A(s)$. Hence and by Berge maximum theorem (see Aliprantis and Border (1994), theorem 17.31) we obtain $\varphi_n \to h^*$.

Step 3. Since λ has Strong Feller Property p^* must be continuous, as: $p^*(s) = \int\limits_K v(c^*(s'), l^*(s'))\lambda(ds'|s)$. Note that

$$
h^*(s) = \arg\max_{(c,l)\in A(s)} H(c, l; p^*, s).
$$

and $s \to H(c, l; p, s)$ is continuous since p^* is. Hence conditions of Berge maximum theorem are satisfied and hence $h^*(\cdot)$ is continuous.

Note that by theorem 2, $\varphi_{2n} \to h^*$ and $\varphi_{2n-1} \to h^*$ and both sequences are monotone, since BR is decreasing operator. Since K is compact, hence both subsequences satisfy condition of Dini Theorem and we obtain uniform continuity of φ_n.

7.3. Proofs in the model without absorbing state

We now turn to a transition without an absorbing state (see assumption 3). By assumptions 1 and 3 the objective becomes:

$$
U(c, l; h, s) := u(c, l) + \beta(h, s) g(s - c, 1 - l) + \gamma(h, s),
$$

with $\beta(h,s) := \int\limits_K v(h_1(y),h_2(y))\lambda_1(dy|s) - \int\limits_K v(h_1(y),h_2(y))\lambda_2(dy|s)$, and $\gamma(h,s) := \int\limits_K v(h_1(y),h_2(y))\lambda_2(dy|s)$.

Define $G(c,l;\beta,\gamma,s) := u(c,l) + \beta g(s-c,1-l) + \gamma$ with $\beta \in \mathbb{R}$ and $\gamma \in \mathbb{R}_+$. We start with some preliminary lemma.

Lemma 11. *For each $\beta \in \mathbb{R}$, $\gamma \in \mathbb{R}_+$ and $s \in K$ the function $(c,l) \to G(c,l;\beta,\gamma,s)$ has a unique maximum.*

Proof. Since $G(\cdot,\cdot;\beta,\gamma,s)$ is continuous on $A(s)$, hence the set of maximization problem must be nonempty. We show that optimal solution is unique. If $\beta > 0$ by Assumption 3 we obtain uniqueness of optimal solution since $(c,l) \to G(c,l;\beta,\gamma,s)$ is strictly concave. If $\beta < 0$ by Assumption 3 we obtain unique solution as well, moreover it is $(s,1)$.

Lemma 12. *Let $\beta_n \to \beta$ and $\gamma_n \to \gamma$. Let $h^n(s) = \arg\max_{(c,l)\in A(s)} G(c,l;\beta_n,\gamma_n,s)$ and $h(s) = \arg\max_{(c,l)\in A(s)} G(c,l;\beta,\gamma,s)$. Then $h^n \to h$ pointwise.*

Proof. Let $\beta_n \leq 0$ and $\beta \leq 0$. Then $h^n \equiv (s,1) = h$. If Let $\beta_n \geq 0$ and $\beta \geq 0$ then $(c,l) \to G(c,l;\beta_n,\gamma_n,s)$ and $(c,l) \to G(c,l;\beta_n,\gamma_n,s)$ are strictly concave on $A(s)$ Moreover, G is continuous with respect to (β,γ). Hence conditions of Berge maximum theorem are satisfied and desired convergence hold.

Proof (of Theorem 4). By lemma 11 and assumption 1 we immediately obtain that there is unique optimal solution of maximization problem of $U(c,l;\bar{h},s)$. Hence we have shown that $\mathcal{BR} : \mathcal{D} \to \mathcal{D}$ is well defined. Moreover, the image of $\mathcal{BR}$ is contained in D i.e. $\mathcal{BR}(\mathcal{D}) \subset D$. Now we show that $\mathcal{BR}$ is continuous in the weak-star topology. Let $\bar{h}_n \to \bar{h}$ in the weak star topology. Note that if $a = (c,l)$ then for each $s \in K$ the function

$$w_k(s',a) := v(a)\rho_k(s',s)$$

is a Caratheodory function. Hence

$$\beta(\bar{h}_n,s) \to \beta(\bar{h},s) \quad \text{as } n \to \infty,$$

and

$$\gamma(\bar{h}_n,s) \to \gamma(\bar{h},s) \quad \text{as } n \to \infty,$$

and hence $U(c,l;\bar{h}_n,s) \to U(c,l;\bar{h},s)$. By Lemma 12 optimal solution of $U(c,l;\bar{h}_n,s)$ must converge to the optimal solution of $U(c,l;\bar{h},s)$. Therefore $\mathcal{BR}(\bar{h}_n) \to \mathcal{BR}(\bar{h})$ pointwise and hence in the weak star topology. Therefore by Schauder-Tikhonov Theorem we conclude that there exists fixed point $h^* = \mathcal{BR}(h^*)$ μ a.e. Let $h^o := \mathcal{BR}(h^*)$ pointwise. Since $\mathcal{BR} : \mathcal{D} \to D$, hence h^o must be a stationary strategy. Since $h^o = h^*$ μ a.e. by definition of the functions $\beta(h,s)$ and $\gamma(h,s)$ we conclude that $\beta(h^*,s) = \beta(h^o,s)$ and $\gamma(h^*,s) = \gamma(h^o,s)$ for each $s \in K$ and hence for each (c,l) we have $U(c,l;h^o,s) = U(c,l;h^*,l)$. Hence $h^o = h^*$ for each $s \in K$ and $h^*(s) = \mathcal{BR}(h^*)(s)$ for each $s \in K$.

Finally to show continuity of a MPNE, one follows the reasoning in the proof of corollary 1.

7.4. Proofs for Stationary Markov Equilibrium

To apply result of Santos and Peralta-Alva (2005), we need to reformulate our specification of the stochastic technology (see also Remark 1). The evolution of the state generated by MPNE $(c^*(\cdot), l^*(\cdot))$ can be recursively written as

$$s_{t+1} \sim g(s_t - c^*(s_t), 1 - l^*(s_t))\lambda_1(\cdot|s_t) + (1 - g(s_t - c^*(s_t), 1 - l^*(s_t)))\lambda_2(\cdot|s_t). \tag{18}$$

This can be alternatively expressed by:

$$s_{t+1} = J(s_t - c^*(s_t), 1 - l^*(s_t), s_t, z_t), \tag{19}$$

where $\{z_t\}_{t=1}^{\infty}$ is an exogenous shock. Mathematically, z_t is a sequence of independent random variables, where each z_t is distributed uniformly on the square $[0,1]^2$, with $J : K \times L \times K \times [0,1]^2 \to K$ a production function of the form:

$$J(i,l,s,z_1,z_2) = \begin{cases} X(z_2,s) & \text{if} \quad z_1 \leq g(i,l), \\ Y(z_2,s) & \text{if} \quad z_1 \geq g(i,l), \end{cases}$$

where for all $s \in K$ $X(\cdot,s)$ is a random variable with the distribution $\lambda_1(\cdot|s)$ and $Y(\cdot,s)$ is a random variable with the distribution $\lambda_2(\cdot|s)$. The next lemma shows that Markov chains described by the transition distribution in (18) and (19) are equivalent, i.e. have the same transition probability.

Lemma 13. *Under Assumptions 1 and 3 the equality holds:*

$$\int_{[0,1]^2} f(J(s - c^*(s), 1 - l^*(s), s, z_1, z_2))dz_1 dz_2 =$$

$$\tilde{g}(s) \int f(s')\lambda_1(ds'|s) + (1 - \tilde{g}(s)) \int f(s')\lambda_2(ds'|s) \qquad (20)$$

for any Borel measurable function f. In the other words, the transition probabilities generated by the distribution in (18) and function in (19) are the same.

Proof. Let f be an arbitrary Borel measurable function from K to $\mathbb{R}$. Put $\tilde{g}(s) = g(s - c^*(s), 1 - l^*(s))$. Then for arbitrary s we have:

$$\int_{[0,1]^2} f(J(s - c^*(s), 1 - l^*(s), s, z_1, z_2))dz_1 dz_2 =$$

$$\tilde{g}(s) \int_0^1 f(X(z_2, s))dz_2 + (1 - \tilde{g}(s)) \int_0^1 f(Y(z_2, s))dz_2 =$$

$$\tilde{g}(s) \int f(s')\lambda_1(ds'|s) + (1 - \tilde{g}(s)) \int f(s')\lambda_2(ds'|s)$$

where the first equality follows by definition of J and the last equality we obtain since $X(\cdot, s)$ and $Y(\cdot, s)$ have distributions $\lambda_1(\cdot|s)$ and $\lambda_2(\cdot|s)$ respectively. The equality in (20) is hence satisfied. As a result the transition probabilities in (18) and (19) are the same.

Proof (of theorem 5). Define $\tilde{J}(s, z_1, z_2) := J(s - c^*(s), 1 - l^*(s), s, z_1, z_2)$ Since the state space is compact we just need to show that Assumption 2 in Santos and Peralta-Alva (2005) is satisfied. That is

$$\phi_f(s) := \int_{[0,1]^2} f(\tilde{J}(s, z_1, z_2))dz_1 dz_2,$$

is a continuous function whenever f is. From Lemma 13, Feller property of λ_1, and λ_2 and continuity of c^* and l^*, we immediately obtain the continuity of ϕ_f. Hence and by Theorem 1 in Santos and Peralta-Alva (2005) there exists an invariant distribution.

Proof (of theorem 6). Let MPNE (c^*, l^*) be given. For a transition probability $Q(\cdot|s-c^*(s), 1-l^*(s), s)$ let us define a corresponding Markov operator $H : \mathcal{C}(K) \to \mathcal{C}(K)$ as

$$H(f)(s) := g(s-c^*(s), 1-l^*(s)) \int_K f(s')\lambda_1(ds'|s)$$
$$+(1-g(s-c^*(s), 1-l^*(s))) \int_K f(s')\lambda_2(ds').$$

Observe that operator H is stable hence $Q(\cdot|s-c^*(s), 1-l^*(s), s)$ has a Feller property. We now show that H is also quasi-compact[19]. To see that let us also define an operator L:

$$L(f)(s) := (1-g(s-c^*(s), 1-l^*(s))) \int_K f(s')\lambda_2(ds'),$$

in $\mathcal{C}(K)$. Endow $\mathcal{C}(K)$ with the topology of uniform convergence and denote a unit ball in $\mathcal{C}(K)$ by $\mathcal{B}$. Note that:

$$L(\mathcal{B}) = \left\{ (1-g(s-c^*(s), 1-l^*(s))) \int_K f(s')\lambda_2(ds') : f \in \mathcal{B} \right\}.$$

Note that

$$L(\mathcal{B}) = \left\{ (1-g(s-c^*(s), 1-l^*(s)))\, \alpha : \alpha \in [0,1] \right\},$$

is the compact set. Hence L is a compact operator. Then:

$$\begin{aligned} |H(f)(s) - L(f)(s)| &= \left| g(s-c^*(s), 1-l^*(s)) \int_K f(s')\lambda_1(ds'|s) \right| \\ &\leq g(s-c^*(s), 1-l^*(s)) \int_K |f(s')|\, \lambda_1(ds'|s) \\ &\leq \sup_{s\in K} g(s,1) < 1. \end{aligned}$$

[19]Endow $\mathcal{C}(K)$ with the topology of uniform convergence. An operator $H : \mathcal{C}(K) \to \mathcal{C}(K)$ is said to be quasi-compact if there exists a natural number n and a compact operator L such that $||H^n - L|| < 1$.

This completes that H is quasi-compact. Finally applying theorem 3.3 from Futia (1982) we get that H is equicontinuous. We take arbitrary element $s_0 \in \text{supp}(\lambda_2)$. Since $Int(\text{supp}(\lambda_2))$ is dense in itself we obtain $U_\varepsilon := (s_0 - \varepsilon, s_0 + \varepsilon) \cap Int(\text{supp}(\lambda_2)) \neq \emptyset$, for all ε. Hence $Q(U_\varepsilon | s - c^*(s), 1 - l^*(s), s) > 0$. Hence Q satisfies uniqueness criterion 2.11 in Futia (1982). Therefore thesis of this theorem follows directly from his theorem 2.12.

Let

$$\mathcal{F} := \{\varphi : S \times [0,1]^2 \to \mathbb{R} : \varphi(s, z_1, z_2) \text{ is continuous in } s \text{ and Borel measurable in } (z_1, z_2)\}.$$

Endow $\mathcal{F}$ with a norm

$$||\varphi|| := \max_{s \in S} \int_{[0,1]^2} |\varphi(s, z_1, z_2)| dz_1 dz_2.$$

Convergence of a sequence in this norm is denoted by $\stackrel{n}{\to}$.

Lemma 14. *Let $c_n \to c$ and $l_n \to l$ uniformly on S. Let $\varphi_n(s, z_1, z_2) := J(s - c_n(s), 1 - l_n(s), s, z_1, z_2)$ and $\varphi(s, z_1, z_2) := J(s - c(s), 1 - l(s), s, z_1, z_2)$. Then $\varphi_n \stackrel{n}{\to} \varphi$.*

Proof. Let $\mathcal{L}$ denote a Lebesgue measure on the real line. Let $E_n^1(s) := \{z_1 : z_1 \leq g(s - c(s), 1 - l(s))\} \cap \{z_1 : z_1 \leq g(s - c_n(s), 1 - l_n(s))\}$, $E_n^2(s) := \{z_1 : z_1 \leq g(s - c(s), 1 - l(s))\} \cap \{z_1 : z_1 > g(s - c_n(s), 1 - l_n(s))\}$, $E_n^3(s) := \{z_1 : z_1 > g(s - c(s), 1 - l(s))\} \cap \{z_1 : z_1 \leq g(s - c_n(s), 1 - l_n(s))\}$ and $E_n^4(s) := \{z_1 : z_1 > g(s - c(s), 1 - l(s))\} \cap \{z_1 : z_1 > g(s - c_n(s), 1 - l_n(s))\}$. Observe that on E_n^1 hold $\varphi_n(s, z_1, z_2) = \varphi(s, z_1, z_2) = X(z_2, s)$ hence

$$\int_{E_n^1(s)} \left(\int_0^1 |\varphi_n(s, z_1, z_2) - \varphi(s, z_1, z_2)| dz_2 \right) dz_1 = 0. \tag{21}$$

Similarly, on $E_n^4(s)$ the following holds $\varphi_n(s, z_1, z_2) = \varphi(s, z_1, z_2) = Y(z_2, s)$ hence

$$\int_{E_n^4(s)} \left(\int_0^1 |\varphi_n(s, z_1, z_2) - \varphi(s, z_1, z_2)| dz_2 \right) dz_1 = 0. \tag{22}$$

Next on $E_n^2(s)$, the following holds $\varphi_n(s, z_1, z_2) = Y(z_2, s)$ and $\varphi(s, z_1, z_2) = X(z_2, s)$ and

$$\begin{aligned}
&\int\limits_{E_n^2(s)} \left(\int\limits_0^1 |\varphi_n(s, z_1, z_2) - \varphi(s, z_1, z_2)| dz_2 \right) dz_1, \\
&= \int\limits_{E_n^2(s)} \left(\int\limits_0^1 |X(z_2, s) - Y(z_2, s)| dz_2 \right) dz_1, \\
&\leq 2S\mathcal{L}(E_n^2(s)).
\end{aligned} \tag{23}$$

Similarly

$$\int\limits_{E_n^3(s)} \left(\int\limits_0^1 |\varphi_n(s, z_1, z_2) - \varphi(s, z_1, z_2)| dz_2 \right) dz_1 \leq 2S\mathcal{L}(E_n^3(s)). \tag{24}$$

Combining (21), (22), (23) and (24) we have

$$\int |\varphi_n(s, z_1, z_2) - \varphi(s, z_1, z_2)| dz_1 dz_2 \leq 2S(\mathcal{L}(E_n^3(s)) + \mathcal{L}(E_n^2(s))). \tag{25}$$

Observe that $E_n^2(s) = \{z_1 : g(s - c_n(s), 1 - l_n(s)) \leq z_1 \leq g(s - c(s), 1 - l(s))\} = [g(s - c_n(s), 1 - l_n(s)), g(s - c(s), 1 - l(s))]$. Hence

$$\mathcal{L}(E_n^2(s)) = |g(s - c(s), 1 - l(s)) - g(s - c_n(s), 1 - l_n(s))|$$

or it is an empty set. Since $c_n \to c$ and $l_n \to l$ uniformly in K and g is uniformly continuous on $K \times L$ hence $\mathcal{L}(E_n^2(s)) \to 0$ uniformly in s. Similarly we show that $\mathcal{L}(E_n^3(s)) \to 0$ uniformly in s. As a result (25) yields desired convergence.

Proof (of Theorem 7). From Lemma 14 we conclude that $J_n \stackrel{n}{\to} J$. The rest follows from Theorem 2 in Santos and Peralta-Alva (2005).

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