A typical macro model with rational expectations can be written as follows:

\[ A_1 \begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A_0 \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_{t+1} \]  

(1)

where:

- \( X_t \): vector \( n \times 1 \), collecting variables predetermined at time \( t \), including shock processes (usually of an AR(1) type), and other backward-looking variables; these variables are often referred to as states
- \( P_t \): vector \( m \times 1 \), collecting forward-looking variables, i.e. those entering the model in form of expectations; these variables are often referred to as jumpers
- \( Z_t \): vector \( k \times 1 \), collecting shocks (white noise)
- \( A_1, A_0 \): matrices \((n + m) \times (n + m)\)
- \( \gamma \): matrix \((n + m) \times k\)

Let us assume for a while that \( A_1 \) is nonsingular (invertible).\(^1\) Then, system (1) can be rewritten as:

\[ \begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + R Z_{t+1} \]  

(2)

where: \( A = A_1^{-1} A_0 \) and \( R = A_1^{-1} \gamma \).

Using the Jordan decomposition, \( A \) can be decomposed as follows:

\[ A = C A C^{-1} \]

where:

- \( \Lambda \): diagonal matrix with eigenvalues of \( A \) on its main diagonal, sorted with increasing absolute value; henceforth we will assume that \( A \) has \( n + m \) distinct eigenvalues
- \( C \): corresponding matrix of eigenvectors

Premultiplying (2) by \( C^{-1} \) yields:

\[ C^{-1} \begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = \Lambda C^{-1} \begin{bmatrix} X_t \\ P_t \end{bmatrix} + C^{-1} R Z_{t+1} \]  

(3)

Let us now write \( \Lambda \) in block form:

\(^1\)Singularity of \( A_1 \) can often be avoided by reducing the number of equations (and hence endogenous variables) so that (1) does not include static relationships (i.e. consisting of current period variables only).
\[
\Lambda = \begin{bmatrix}
\Lambda_1 & 0 \\
0 & \Lambda_2
\end{bmatrix}
\]

where:
\(\Lambda_1\): matrix \(n^* \times n^*\), collecting eigenvalues lying within the unit circle (i.e. the modulus smaller or equal to unity)
\(\Lambda_2\): matrix \(m^* \times m^*\), collecting eigenvalues outside the unit circle

Similarly for further use, let us decompose \(A\), \(C\) and \(R\):

\[
A = \begin{bmatrix}
A_{11[n\times n]} & A_{12[n\times m]} \\
A_{21[m\times n]} & A_{22[m\times m]}
\end{bmatrix} \quad C^{-1} = \begin{bmatrix}
C_{11[n^*\times n]} & C_{12[n^*\times m]} \\
C_{21[m^*\times n]} & C_{22[m^*\times m]}
\end{bmatrix} \quad R = \begin{bmatrix}
R_{1[n\times k]} \\
R_{2[m\times k]}
\end{bmatrix}
\]

Now we define transformed variables \(Y_t[n^*\times 1]\) and \(Q_t[m^*\times 1]\):

\[
\begin{bmatrix}
Y_t \\
Q_t
\end{bmatrix} = C^{-1} \begin{bmatrix}
-X_t \\
P_t
\end{bmatrix}
\]

which allows us to rewrite (3) as:

\[
\tilde{E}_t \begin{bmatrix}
Y_{t+1} \\
Q_{t+1}
\end{bmatrix} = \Lambda \begin{bmatrix}
Y_t \\
Q_t
\end{bmatrix} + C^{-1} R Z_{t+1}
\]

where \(\tilde{E}_t\) is modified expectations operator such that \(\tilde{E}_t X_{t+1} = X_{t+1}\) and \(\tilde{E}_t P_{t+1} = E_t P_{t+1}\).

Iterating the bottom blocks of (5) forward, applying expectations operator \(E_t\) and using \(E_t Z_{t+j+1} = 0\) yields (for \(j \geq 0\)):

\[
E_t Q_{t+j+1} = \Lambda^j_2 Q_t
\]

Note that eigenvalues collected in \(\Lambda_2\) lie outside the unit circle. Then, equation (6) is nonexploding if and only if \(Q_t = 0\forall t\). Using (4) to go back to our original variables and using the block structure of \(C^{-1}\), this implies:

\[
C_{21} X_t + C_{22} P_t = 0
\]

**Theorem 1:** If the following conditions are satisfied:

(i) the number of eigenvalues of \(A\) lying outside the unit circle is equal to the number of jumpers, i.e. \(m^* = m\) (Blanchard-Kahn condition)

(ii) \(C_{22}\) is of full rank (rank condition)

then there exist a unique solution to (2).

The proof can be presented as follows. If both conditions of Theorem 1 hold, (7) allows us to determine uniquely the value of jumpers for any given states:

\[
P_t = -C_{22}^{-1} C_{21} X_t
\]
Moving back to (2), the law of motion for state variables can be written as:

\[ X_{t+1} = (A_{11} - A_{12}C_{22}^{-1}C_{21})X_t + R_tZ_{t+1} \]  

(9)

Hence, we have the full and unique solution to the original dynamic system (2). The solution has a recursive form, given by (8) and (9).

***

What if the conditions of Theorem 1 are not satisfied?

Theorem 2: If the number of eigenvalues of \( A \) lying outside the unit circle is larger than the number of jumpers, i.e. \( m^* > m \), or \( m^* = m \) but \( C_{22} \) is rank deficient, then there exist no stable solution to (2).

Proof: note that (7) imposes (for given \( X_t \)) \( m^* \) restrictions on \( m \) elements of \( P_t \). Hence, for \( m^* > m \) a stable (nonexploding) solution does not exist. Similarly, if \( C_{22} \) is square but rank deficient, system (7) does not have a solution.

Theorem 3: If the number of eigenvalues of \( A \) lying outside the unit circle is smaller than the number of jumpers, i.e. \( m^* < m \), then there exist infinitely many solutions to (2).

Proof: As in the case of Theorem 2, this time there exist infinitely many vectors \( P_t \) that satisfy (7) for any \( X_t \). In particular, the realized value of \( P_t \) may depend on information outside of the model, i.e. shocks not included in \( Z_t \) (so-called sunspot shocks) may affect the solution.

***

The algorithm presented above relies on transforming the original problem (1) into (2), which requires invertibility of \( A_1 \). What if this condition does not hold? The solution is to use the Schur decomposition (so-called QZ decomposition), which allows to write (1) as:

\[ Q'\Lambda Z' \begin{bmatrix} X_{t+1} \\ E_tP_{t+1} \end{bmatrix} = Q'\Omega Z' \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_{t+1} \]  

(10)

where \( \Omega \) and \( \Lambda \) are upper triangle matrices, while \( Q \) and \( Z \) are unitary matrices (\( QQ' = I \), \( ZZ' = I \)).

The ordering is now according to the generalized eigenvalues \( \frac{\omega_{ii}}{\lambda_{ii}} \), where \( \omega_{ii} \) and \( \lambda_{ii} \) are \( i \)-th elements of the main diagonals of \( \Omega \) and \( \Lambda \).

Following the same steps as in the original Blanchard-Kahn algorithm we obtain:

\[ \tilde{E}_t \begin{bmatrix} Y_{t+1} \\ Q_{t+1} \end{bmatrix} = \Lambda^{-1}\Omega \begin{bmatrix} Y_t \\ Q_t \end{bmatrix} + \Lambda^{-1}Q\gamma Z_{t+1} \]  

(11)
Since $\Lambda^{-1}\Omega$ is triangular, system (11) has a recursive form and can be solved as presented before.

The algorithm using the QZ decomposition can be implemented with routine \textit{gensys} by C. Sims, coded for Matlab/Octave and R. Documentation and codes can be downloaded from: \url{http://sims.princeton.edu/yftp/gensys/}.  
