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Solving DSGE models using the Blanchard-Kahn algorithm (with extensions)

A typical macro model with rational expectations can be written as follows:

$$A_1 \begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A_0 \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_{t+1} \tag{1}$$

where:
X_t: vector $n \times 1$, collecting variables predetermined at time t, including shock processes (usually of an AR(1) type), and other backward-looking variables; these variables are often referred to as states
P_t: vector $m \times 1$, collecting forward-looking variables, i.e. those entering the model in form of expectations; these variables are often referred to as jumpers
Z_t: vector $k \times 1$, collecting shocks (white noise)
A_1, A_0: matrices $(n+m) \times (n+m)$
γ: matrix $(n+m) \times k$

Let us assume for a while that A_1 is nonsingular (invertible).[1] Then, system (1) can be rewritten as:

$$\begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = A \begin{bmatrix} X_t \\ P_t \end{bmatrix} + R Z_{t+1} \tag{2}$$

where: $A = A_1^{-1} A_0$ and $R = A_1^{-1} \gamma$.

Using the Jordan decomposition, A can be decomposed as follows:

$$A = C \Lambda C^{-1}$$

where:
Λ: diagonal matrix with eigenvalues of A on its main diagonal, sorted with increasing absolute value; henceforth we will assume that A has $n+m$ distinct eigenvalues
C: corresponding matrix of eigenvectors

Premultiplying (2) by C^{-1} yields:

$$C^{-1} \begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = \Lambda C^{-1} \begin{bmatrix} X_t \\ P_t \end{bmatrix} + C^{-1} R Z_{t+1} \tag{3}$$

Let us now write Λ in block form:

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[1] Singularity of A_1 can often be avoided by reducing the number of equations (and hence endogenous variables) so that (1) does not include static relationships (i.e. consisting of current period variables only).

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{bmatrix}$$

where:
Λ_1: matrix $n^* \times n^*$, collecting eigenvalues lying within the unit circle (i.e. the modulus smaller or equal to unity)
Λ_2: matrix $m^* \times m^*$, collecting eigenvalues outside the unit circle

Similarly for further use, let us decompose A, C^{-1} and R:

$$A = \begin{bmatrix} A_{11[n\times n]} & A_{12[n\times m]} \\ A_{21[m\times n]} & A_{22[m\times m]} \end{bmatrix} \qquad C^{-1} = \begin{bmatrix} C_{11[n^*\times n]} & C_{12[n^*\times m]} \\ C_{21[m^*\times n]} & C_{22[m^*\times m]} \end{bmatrix} \qquad R = \begin{bmatrix} R_{1[n\times k]} \\ R_{2[m\times k]} \end{bmatrix}$$

Now we define transformed variables $Y_{t[n^*\times 1]}$ and $Q_{t[m^*\times 1]}$:

$$\begin{bmatrix} Y_t \\ Q_t \end{bmatrix} = C^{-1} \begin{bmatrix} X_t \\ P_t \end{bmatrix} \tag{4}$$

which allows us to rewrite (3) as:

$$\tilde{E}_t \begin{bmatrix} Y_{t+1} \\ Q_{t+1} \end{bmatrix} = \Lambda \begin{bmatrix} Y_t \\ Q_t \end{bmatrix} + C^{-1} R Z_{t+1} \tag{5}$$

where $\tilde{E}_t$ is modified expectations operator such that $\tilde{E}_t X_{t+1} = X_{t+1}$ and $\tilde{E}_t P_{t+1} = E_t P_{t+1}$.

Iterating the bottom blocks of (5) forward, applying expectations operator E_t and using $E_t Z_{t+j+1} = 0$ yields (for $j \geq 0$):

$$E_t Q_{t+j+1} = \Lambda_2^j Q_t \tag{6}$$

Note that eigenvalues collected in Λ_2 lie outside the unit circle. Then, equation (6) is nonexploding if and only if $Q_t = 0 \forall t$. Using (4) to go back to our original variables and using the block structure of C^{-1}, this implies:

$$C_{21} X_t + C_{22} P_t = 0 \tag{7}$$

Theorem 1: If the following conditions are satisfied:
(i) *the number of eigenvalues of A lying outside the unit circle is equal to the number of jumpers, i.e. $m^* = m$ (Blanchard-Kahn condition)*
(ii) *C_{22} is of full rank (rank condition)*
then there exist a unique solution to (2).

The proof can be presented as follows. If both conditions of Theorem 1 hold, (7) allows us to determine uniquely the value of jumpers for any given states:

$$P_t = -C_{22}^{-1} C_{21} X_t \tag{8}$$

Moving back to (2), the law of motion for state variables can be written as:

$$X_{t+1} = \left(A_{11} - A_{12}C_{22}^{-1}C_{21}\right) X_t + R_1 Z_{t+1} \tag{9}$$

Hence, we have the full and unique solution to the original dynamic system (2). The solution has a recursive form, given by (8) and (9).

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What if the conditions of Theorem 1 are not satisfied?

Theorem 2: If the number of eigenvalues of A lying outside the unit circle is larger than the number of jumpers, i.e. $m^* > m$, or $m^* = m$ but C_{22} is rank deficient, then there exist no stable solution to (2).

Proof: note that (7) imposes (for given X_t) m^* restrictions on m elements of P_t. Hence, for $m^* > m$ a stable (nonexploding) solution does not exist. Similarly, if C_{22} is square but rank deficient, system (7) does not have a solution.

Theorem 3: If the number of eigenvalues of A lying outside the unit circle is smaller than the number of jumpers, i.e. $m^* < m$, then there exist infinitely many solutions to (2).

Proof: As in the case of Theorem 2, this time there exist infinitely many vectors P_t that satisfy (7) for any X_t. In particular, the realized value of P_t may depend on information outside of the model, i.e. shocks not included in Z_t (so-called sunspot shocks) may affect the solution.

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The algorithm presented above relies on transforming the original problem (1) into (2), which requires invertibility of A_1. What if this condition does not hold? The solution is to use the Schur decomposition (so-called QZ decompostion), which allows to write (1) as:

$$Q'\Lambda Z' \begin{bmatrix} X_{t+1} \\ E_t P_{t+1} \end{bmatrix} = Q'\Omega Z' \begin{bmatrix} X_t \\ P_t \end{bmatrix} + \gamma Z_{t+1} \tag{10}$$

where Ω and Λ are upper triangle matrices, while Q and Z are unitary matrices ($QQ' = I$, $ZZ' = I$).
The ordering is now according to the generalized eigenvalues $\frac{\omega_{ii}}{\lambda_{ii}}$, where ω_{ii} and λ_{ii} are i-th elements of the main diagonals of Ω and Λ.

Following the same steps as in the original Blanchard-Kahn algorithm we obtain:

$$\tilde{E}_t \begin{bmatrix} Y_{t+1} \\ Q_{t+1} \end{bmatrix} = \Lambda^{-1}\Omega \begin{bmatrix} Y_t \\ Q_t \end{bmatrix} + \Lambda^{-1}Q\gamma Z_{t+1} \tag{11}$$

Since $\Lambda^{-1}\Omega$ is triangular, system (11) has a recursive form and can be solved as presented before.

The algorithm using the QZ decomposition can be implemented with routine `gensys` by C. Sims, coded for Matlab/Octave and R. Documentation and codes can be downloaded from: `http://sims.princeton.edu/yftp/gensys/`.